# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1435

## ELLIPTIC FUNCTIONS AND INTEGRALS WITH REAL MODULUS IN FLUID MECHANICS

By Robert Legendre

Translation of "Les Fonctions et Intégrales Elliptiques a Module Réel en Mécanique des Fluides." ONERA, Publication No. 71, 1954.

Washington
June 1958

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# ELLIPTIC FUNCTIONS AND INTEGRALS WITH REAL MODULUS IN FLUID MECHANICS*

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## SUMMARY

Advantage of the elliptic functions and of the more general functions of Schwarz for fluid mechanics. Flows outside and inside polygons. Application to the calculation of an elbow diffuser for a wind tunnel. Properties of the elliptic integrals of the first kind and of the elliptic functions. Properties of the theta functions and decomposition of the elliptic functions into products of theta functions. Properties of the zeta functions. Decomposition of the elliptic functions into sums of zeta functions and calculation of the elliptic integrals. Applications to the calculation of wing profiles, of compressor profiles, and to the study of the vibrations of airplane wings and of compressor vanes.

The manuscript of the present paper was checked by Mr. Eichelbrenner who corrected several imperfections and suggested numerous improvements to make reading of the paper easier. However, the limited subject does not permit filling in more than an incomplete knowledge of the properties of analytic functions.

## INTRODUCTION

The solutions of a very large number of problems in fluid mechanics are expressed with the aid of elliptic functions. The mechanism of the role of these functions is not difficult to analyze.

From one point of view, the elliptic functions can be considered as the simplest ones (after the exponential and circular ones which they generalize) among the solutions of differential equations whose coefficients are polynomials. Thus, it is natural that one must resort to elliptic functions when a somewhat close approximation is desired.

From another more geometrical view point, the majority of the problems which can be solved with the aid of exponential and circular functions are related to schemes which assemble the given quantities on a

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NACA Reviewer's note: The original French publication contains certain typographical errors and obvious omissions in equations that have been corrected without comment.

unique segment or else on a unique curve or on a family of curves, deduced one from another by a simple periodicity. As soon as the given parameters are on two separate curves or on a family of curves deduced one from another by a double periodicity, the elliptic functions are introduced.

The elliptic functions are analytic functions, the field of which represents directly the plane flow of an ideal incompressible fluid. The conformal transformations allow the association of the fields of flow around very different obstacles. On the other hand, the study of the flow of compressible fluids is largely based upon the acquired knowledge of analytic functions, at least as far as search for approximations by various artifices is concerned.

Among these artifices one must mention the study of the hodograph, that is, of the potential of the plane flow of a compressible fluid represented with the aid of the velocity. The relation between the potential and the velocity is in fact much closer to the one expressed by an analytic function than the relation between the potential and the coordinates in the physical plane. The theory of conical flows satisfying an equation which is linearized by approximation, on the other hand, leads to the study of the three-dimensional flow of a compressible fluid in terms of the plane flow of an incompressible fluid. Finally, the projection of the velocity of the almost uniform flow of a compressible fluid onto a plane perpendicular to the mean direction of this velocity is approximately the velocity of the flow of an incompressible fluid. The nearer the Mach number of the mean flow is to unity, the closer is the approximation.

The theory of the elliptic functions is generally very little known among engineers, not so much for want of mathematical treatises as because of the lack of work establishing the connection between these functions and the flows which they can represent. One finds quite often a thoroughly documented treatise which does not contain a single figure.

The present paper uses largely the geometrical methods taught in 1930 at the Ecole d'Application du Génie Maritime by the Institute member, Mr. Emile Barrillon. It does not attempt the rigorousness to be found in the mathematical treatments but the intuitive and quick solution of the problems. The general theorems relative to the analytic functions permit a justification of the exactness and uniqueness of the solutions found.

Although the paper belongs in the domain of applied mathematics, it establishes some properties of the elliptic functions which are, to our knowledge, original.

# 1. SCHWARZ' TRANSFORMATION

## 1.0 Flow Outside or Inside a Polygon

The later developments will show that the elliptic functions with real modulus and the integrals which can be attached to them represent flows outside and inside quadrirectangular polygons. They generalize the circular functions which are associated with flows limited by birectangular polygons and are themselves particular cases of the Schwarz functions defining the flows outside or inside arbitrary polygons. Finally, Schwarz' functions are the simplest of the automorphic (or Fuchsian) functions of Poincaré.

Without aspiring to such large generalizations for the limited subject of the present paper, although the geometrical methods are of a character to facilitate the understanding and to generalize the application of Poincaré's functions, it is useful to associate the elliptic functions with real modulus to the functions of Schwarz.

Let us recall the principal properties of the analytic functions which will be useful later on.

An analytic function is associated with a representative field constituted by a net of two families of orthogonal curves forming a grid. Each curve corresponds to a constant value of the real part or of the imaginary part of the function. For the interpretation in terms of fluid mechanics, one of the families corresponds to the equipotentials, and the other to the stream lines.

An analytic function of a variable, which itself is an analytic function of a second variable, is an analytic function of the second variable. Its representative field in the plane of the second variable may be constructed by a point-by-point conformal correspondence, that is, locally conserving the angles and the ratios of the lengths except of the second order. An analytic function such as the one above which defines the correspondence between the two variables is thus associated with a conformal geometrical transformation.

An analytic function defined in the entire plane of a variable differs only by a constant from the sum of its principal parts in the neighborhood of all its singular points. If an analytic function is defined in a limited region of the plane, but if it is real or has a constant imaginary part on the contour of this region, and if it is possible to make this region correspond, by conformal transformation, to a half plane or to the interior of a circle in such a manner that the function can be defined by analytic continuation in the entire plane of the new variable, it differs only by a constant from the sum of its principal

parts in the neighborhood of all its singularities and from the singularities of the analytic continuation in the entire plane of the new variable.

### 1.1 Hodograph of the Flow Outside or Inside a Polygon

In accordance with the foregoing review, it suffices to characterize one flow defining a conformal transformation of the polygon into a circle or into a straight line in order to determine all flows containing doublets, sources, sinks, or vortices in limited number or distributed on the curves in infinite number. These flows will, in fact, be associated with an analytic function, determined except for one constant for these enumerated singularities and their images, by inversion with respect to the circle or by symmetry with respect to the straight line.

For reasons of symmetry, the retained fundamental flow will be that of a circulation around the polygon or that of an isolated vortex inside the polygon.

Let $u$ be the complex variable in the plane of the polygon and $\chi(u)$ the complex potential of the flow whose real part is the potential of the velocities, while a streamline corresponds to a constant value of the imaginary part. We can select $\chi$ real on the contour of the polygon (figs. 1 and 2).

The velocity has a constant direction on every side of the polygon. This property may be easily characterized by use of the hodograph, that is to say, of the velocity or also of its inverse, studied as a function of $\chi$

$$\frac{du}{d\chi} = \rho e^{i\theta} = \zeta(\chi)$$

where $\rho$ is the inverse of the intensity of the velocity, and $\theta$ the angle of inclination of the velocity, constant on one side of the polygon.

Even more convenient will be the study of the logarithmic hodograph

$$\ln\zeta = \ln\rho + i\theta$$

the imaginary part of which is a constant on one side of the polygon corresponding, in addition, to the real axis of the $\chi$- plane or to the circle of radius 1 in the plane of the variable $z = e^{i\chi}$. The variable $z$ will always be associated with the function $\chi$ in the following calculations. Comparison of the figures 1 or 2 with figure 1(a) shows, in fact, that the conformal transformation $u(z)$ defined implicitly for $\chi(u)$ and $\chi(z)$

makes the exterior or interior of a polygon correspond to the interior of a circle. Another transformation $u(t)$, defined further on, will make the exterior or interior of a polygon correspond to a half plane.

## 1.2 General Solution

Let us choose to study $\ln\zeta$ as a function of $z = e^{i\chi}$. The vertices of the polygon correspond to the unknown points of the unit circle $z_1, z_2. . ., z_k. . ., z_n$, which are singular points of the function $u(z)$. No other singular point exists, except perhaps at the origin, center of the circle, which we agree to make correspond to the point at infinity of the field outside the polygon or to the point of reference of the field inside the polygon since the desired circulation has no other singular point outside the polygon but the point at infinity in the case of external flow, and the center of the chosen vortex in the case of internal flow.

The constancy of the imaginary part of $\ln\zeta$ on the unit circle in the plane of the variable $z$ permits defining the analytic continuation of the function toward the outside of the circle. It is sufficient to make correspond to $z$ and $\ln\zeta$, for an internal point, the relation between $1/\bar{z}$ and $\ln\bar{\zeta} + C^{te}$ defining an analytic function at the external point which is the inverse of the first with respect to the circle. The two functions join on the circle where $z = 1/\bar{z}$ and

$$\ln\bar{\zeta} = \ln\rho - i\theta$$

It suffices to adopt as constant $2i\theta$, that is to say, the angle of inclination of the side of the polygon multiplied by $2i$, arranging cuts between the singular points and, for instance, the small-straight lines extending the radii (fig. 3).

The mode of analytic continuation shows that there exists no singular point outside the circle, except perhaps at infinity. All singular points are thus known and the function may be defined by the sum of its singularities.

In the proximity of a vertex $u_k$ of the polygon, the function $\chi(u)$ behaves like

$$\chi = \chi_k + C_1\left(u - u_k\right)^{\frac{\pi}{\Theta_k}} + . . .$$

where $\Theta_k$ is the angle at the vertex measured toward the fluid. Consequently,

$$u - u_k = C_2\left(X - X_k\right)^{\frac{\Theta_k}{\pi}} + \ . \ . \ .$$

$$\zeta = \frac{du}{dx} = C_3\left(X - X_k\right)^{\frac{\Theta_k}{\pi} - 1} + \ . \ . \ .$$

Since $z(X) = e^{iX}$ is regular at a vertex $z_k = e^{iX_k}$,

$$\zeta = C_4\left(z - z_k\right)^{\frac{\Theta_k}{\pi} - 1} + \ . \ . \ .$$

$$\ln\zeta = \left(\frac{\Theta_k}{\pi} - 1\right)\log\left(z - z_k\right) + \ . \ . \ .$$

At infinity, for the flow outside a polygon, and around the selected vertex for the flow inside a polygon, the function $X(u)$ behaves like a logarithm

$$X = \pm i \ln u + \ . \ . \ .$$

The plus sign corresponds to a flow outside a polygon and the minus sign to a flow inside one.

$$\frac{1}{\zeta} = \frac{dX}{du} = \pm \frac{i}{u} + \ . \ . \ . = \pm ie^{\mp iX} + \ . \ . \ .$$

$$\ln \zeta = \pm iX + \ . \ . \ . = \mp\ln z$$

It is not necessary to study the infinity of the z-plane where the singularity is determined by the analytic continuation

Finally, $\ln\zeta$ is defined by the sum of its singularities

$$\ln \zeta = \pm\ln z + \Sigma\left(\frac{\Theta_k}{\pi} - 1\right)\ln\left(z - z_k\right) + C^{te}$$

If one notes that the sum of the deviations of the stream on the contour of the polygon is four right angles, then

$$\Sigma\left(\frac{\Theta_k}{\pi} - 1\right) = \pm 2$$

where the plus sign corresponds to the flow outside a polygon. Consequently, the function $\ln\zeta$ behaves, for an infinite $z$, like $\pm\ln z$. On the other hand

$$z - z_k = e^{i\chi} - e^{i\chi_k} = 2ie^{i\frac{\chi+\chi_k}{2}} \sin\frac{\chi - \chi_k}{2}$$

and

$$\ln\zeta = \Sigma\left(\frac{\Theta_k}{\pi} - 1\right)\ln\left(\sin\frac{\chi - \chi_k}{2}\right) + C^{te}$$

$$\zeta = \frac{du}{d\chi} = C\Pi\left(\sin\frac{\chi - \chi_k}{2}\right)^{\frac{\Theta_k}{\pi} - 1}$$

$$u = C\int\Pi\left(\sin\frac{\chi - \chi_k}{2}\right)^{\frac{\Theta_k}{\pi} - 1} d\chi$$

The flow outside or inside a regular polygon with $n$ sides corresponds to real values $\chi_k$ of $\chi$ stepped at $2\pi/n$ and to values of $\Theta_k$ exactly equal to $\pi \pm 2\pi/n$. The plus sign corresponds to the external flow.

## 1.3 Extension to More Complex Polygons

The formula established above for values of $\Theta_k$ between 0 and $2\pi$ can be generalized.

A zero value of $\Theta_k$ corresponds to a polygon vertex at infinity between two parallel sides (fig. 5).

A value of $\Theta_k$ equal to $2\pi$ corresponds to a point of return (fig. 6).

A negative value of $\Theta_k$ corresponds to two infinite branches inclined by $-\Theta_k$ (fig. 7).

A value of $\Theta_k$ higher than $2\pi$ corresponds to an overlapping between the two adjoining sides (fig. 8).

Finally, the formula defines a polygon, for a real $X$, if $\frac{1}{2}\Sigma\left(\frac{\Theta_k}{\pi} - 1\right)$ is a positive or negative integer different from $\pm 1$.

Even when all values of $\Theta_k$ are contained between 0 and $2\pi$, the choice of the values of $X_k$ for the representation of an a priori given polygon is difficult. It is generally expedient to resort to the electrical analogy, and the analytic expression is no longer applicable except to the exact numerical study in the neighborhood of the singularities.

When $X$ varies from 0 to $2m\pi$ where m is the entire value of $\frac{1}{2}\Sigma\left(\frac{\Theta_k}{\pi} - 1\right)$, the variable u resumes its initial value and the polygon closes. In fact, all cuts in the plane of the variable z may be chosen outside the unit circle, and the integral of $\zeta\, dX$ on that circle is equal to the integral around an infinitely small circle. Since $\zeta$ is, except for one factor, equivalent to z, the integral

$$u = \int \zeta\, dX \sim \int \zeta \frac{dz}{iz}$$

is itself equivalent to z, except for one factor, and, since it is independent of the integration contour, it is necessarily zero.

## 1.4 Change of Reference

For the flow inside a polygon and for the flow limited by a polygon having infinite branches, the vortex of reference was chosen arbitrarily.

If another field or reference $X'$ is selected, it may be useful to determine its relation with the initial reference $X$.

In the plane of the variable $z = e^{iX}$, the field $X'$ is that of a vortex inside the unit circle. It is, except for one constant, defined by this singularity situated at a point the complex variable of which will be denoted by $e^{\alpha+i\beta}$ and by the image of the latter with respect to the circle.

It is convenient to make a change of axes which makes the symmetry evident, taking $ze^{-i\beta}$ as the variable. The new function $X'$ then is

$$X' = -i \ln\left(ze^{-i\beta} - e^{-\alpha}\right) + i \ln\left(ze^{-i\beta} - e^{\alpha}\right) + \beta' - i\alpha$$

The imaginary part of the constant $-i\alpha$ was chosen so that $X'$ should be real on the unit circle in the plane of $z$, image of the contour of the polygon.

If one sets $z' = e^{iX'}$, the relation between $z'$ and $z$ is homographic

$$\left(z'e^{-i\beta'}\right)\left(ze^{-i\beta}\right) - e^{\alpha}\left(z'e^{-i\beta'} + ze^{-i\beta}\right) + 1 = 0$$

The relation between $X'$ and $X$ may be put in another form, convenient for the real values

$$\tan\frac{X' - \beta'}{2}\tan\frac{X - \beta}{2} + \tanh\frac{\alpha}{2} = 0$$

This expression shows the advantage of the function

$$t(u,\beta) = \tan\frac{X - \beta}{2} = -i\,\frac{z - e^{i\beta}}{z + e^{i\beta}}$$

which is real on the contour of the polygon and depends on a real parameter $\beta$. The above formula shows that $t(z,\beta)$ is represented in the z-plane by a doublet on the circumference at the point $z = -e^{i\beta}$, assuming the unit circle as a streamline. The conformal transformation $z(u)$ thus makes the function $t(u)$ correspond to $t(z)$; this function $t(u)$ is represented in the field of the variable $u$ by a doublet on a side or at a vertex of the polygon (fig. 10), according to whether the value chosen for $\beta$ does not or does correspond to the vertex of the polygon in the transformation $z(u)$.

When no reason of symmetry makes it advisable to prefer the functions $X$ or $z = e^{iX}$ for defining the conformal transformation of a polygon into a circle, it is often convenient to utilize a transformation $t(u,\beta)$ for a judicious value of $\beta$ which makes the real axis of $t$ correspond to the polygon, and one of the half-planes limited by the real axis to the interior or to the exterior of the polygon. The transformation $t(u,\beta)$ may be defined directly by

$$\frac{du}{dt} = C_5\left(1 + t^2\right)^{-1 - \frac{1}{2}\Sigma\left(\frac{\Theta_k}{\pi} - 1\right)} \Pi\left(t - \tan\frac{X_k - \beta}{2}\right)^{\frac{\Theta_k}{\pi} - 1}$$

where

$$C_5 = 2C\Pi\left(\cos\frac{\beta - X_k}{2}\right)^{\frac{\Theta_k}{\pi} - 1}$$

This expression is suitable only when the doublet of $t(z)$ has not been chosen at a vertex of the polygon. If, on the contrary, $\beta = X_1 + \pi$, the expression maintains the same form but the term corresponding to $X_1$ must be omitted in each of the products.

The change of reference $t = \tanh \frac{\alpha/2}{t'}$ is particularly convenient with the latter expressions. It shows that $\frac{du}{dt'}$ has the same form as $\frac{du}{dt}$ which was obviously necessary.

For the flow inside a polygon, $\Sigma\left(\frac{\Theta_k}{\pi} - 1\right) = -2$ and the term in $1 + t^2$ disappears. In the other cases, the presence of this term indicates the existence of a point at infinity or of a critical point at the center of the vortex $X(u)$.

If all values of $\Theta_k$ are integer multiples of $\Pi$, the expression $\frac{du}{dt}$ is a rational fraction of $t$ and the integral is expressed with the aid of rational fractions and of logarithms, that is, of functions which are the inverse of exponential or circular functions.

If one of the values of $\Theta_k$ is an odd multiple of $\pi/2$, there exists at least one other value of $\Theta_k$ of this type if all others are multiples of $\Pi$, since the sum of the values of $\Theta_k$ is a multiple of $2\pi$. The polygon has then two right angles and $du/dt$ which is a square root of a polynomial of the second degree, multiplied by a rational fraction in $t$, is again integrated with the aid of rational functions and of exponential or circular functions.

If three values of $\Theta_k$ are odd multiples of $\pi/2$, there exists a fourth value of $\Theta_k$ of this type for all others to be multiples of $\pi$. The polygon has then four right angles, and $du/dt$ is the square root of a polynomial of the fourth degree in $t$, multiplied by a rational function. The function $u(t)$ is called an elliptic integral and its properties will be studied in the following chapters.

### 1.5 Field of Doublets or of Sources and Vortices in a Polygon

With the use of a conformal transformation which makes a fraction of the plane limited by a polygon correspond to the interior of a circle or to a half plane, it is easy to study the field $F(u)$ of doublets or of sources and sinks limited by a polygon which is transformed into a field defined in the entire plane by analytic continuation with the aid of images with respect to the circle or with respect to the real axis.

We shall choose to utilize a transformation $t(u)$ for studying the field $F(u)$ which is transformed into $F(t)$ defined in the entire plane by symmetry around the real axis of $t$.

The sum of the singularities of doublets, or of poles of a higher order forms a rational fraction $F_1(t)$ the analytic continuation of which is $\bar{F}_1(t)$, obtained by replacing the coefficients by their conjugates. The field $F(t) = F_1(t) + \bar{F}_1(t)$ is a rational fraction with real coefficients. Inversely, such a rational fraction defines by a transformation $t(u)$ a field of poles $F(u)$ admitting a polygonal streamline.

The singularities corresponding to sources, sinks, and vortices are logarithms. The function $F(u)$ becomes, after the transformation $t(u)$

$$F(t) = \Sigma\left[A \ln(t - tj) + \bar{A} \ln(t - \bar{t}j)\right] + C^{te}$$

Inversely, an expression of this form defines a field of vortex sources which contains, except when $\Sigma(A + \bar{A}) = 0$, a source on the contour of the polygon, for infinite $t$.

A frequently encountered particular case corresponds to the sources and sinks, the strengths of which are equal or are in a simple fractional relationship. The function $F(t)$ is then the logarithm of a rational fraction.

The field of a line of sources or of vortices is defined by

$$F(t) = \int_C A(s)\ln\left(t - t_s\right)ds + \int_{\bar{C}} \bar{A}(s)\ln\left(t - \bar{t}_s\right)d\bar{s}$$

where $A(s)$ is an arbitrary intensity distributed on the line $C$ as a function of a parameter $s$ which can be the curvilinear abscissa. The intensity $A(s)ds$ is real for sources and purely imaginary for vortices.

### 1.6 Analytic Continuation, Periods, Case of Reduction

The analytic continuation of a real function (or one with a constant imaginary part) on each of the sides of a polygon is obtained by successive symmetries with respect to the sides. The continuation thus defined is generally multiform and the procedure followed depends on the order of the symmetries.

The continuation is periodical. Two successive symmetries, one with respect to a side of the polygon, the other with respect to the homologue

of another side in the first symmetry, are equivalent to the combination of a translation and of a rotation by an angle double that of the two sides.

If the symmetries are combined in such a manner that the rotations have as their sum a multiple of $2\pi$ without the translation being zero, the function $F(u)$ admits this translation as the period.

Figure 11 illustrates the construction of a period in the case of a triangle.

The rotation after the sixth symmetry is

$$-2\alpha - 2\beta - 2\gamma = -2\pi$$

For the rectangles and the equilateral triangles, the symmetries show the way, and the sense of description which is inverted after each symmetry, is reproduced after rotation around a vertex. A function $F(u)$, which is real on the contour of the rectangle or of the triangle, is then uniformly continued. For regular hexagons, the symmetries again show the way but two turns must be described around a vertex in order to reproduce the sense of description. We shall find finally that the uniformity of the function defined in the entire plane, combined with the double periodicity, is characteristic of elliptic functions.

The analytic continuation by symmetries permits utilizing, for the study of a symmetrical flow in a polygon, the transformation relative to the half polygon limited by the axis of symmetry. In order to study, for instance, the field $F(u)$ of a doublet on the axis of symmetry of a quadrilateral with a circulation such that the stagnation points are at the vertices (fig. 12(a)), it will be convenient to utilize the transformation $t(u)$ relative to one of the triangles (fig. 12(b)) and continued analytically in the other

$$\frac{du}{dt} = (t - 1)^{\frac{\beta}{\pi} - 1} (t + 1)^{\frac{\alpha}{\pi} - 1}$$

In the plane of the variable $t$, the function $F(t)$ has the appearance represented by figure 12(c). The cuts will be eliminated by the classical transformation $2t = s + 1/s$ and the function $F(s)$, uniform in the entire plane, will be defined by its singularities

$$F(s) = \frac{Ae^{i\omega}}{s - e^{i\omega}} + iB \ln\left(s - e^{i\omega}\right) + \frac{Ae^{-i\omega}}{s - e^{-i\omega}} - iB \ln\left(s - e^{-i\omega}\right) + C$$

The condition of separation of the stream at $C$ in the $F(u)$-plane imposes an infinite branch in the $F(t)$-plane and a stagnation point at

the origin in the $F(s)$-plane, either with $dF/ds$ being zero for $s = 0$ or else $B = -A \cot \omega$. The constants $\omega$, $A$, $C$, remain arbitrary. The first defines the position of the doublet in the $F(u)$-plane, the others fix the scale and the origin of $F$. Finally, the field $F(u)$ to be studied is defined parametrically by $F(s)$ calculated above and $u(t)$ defined by integration of $du/dt$ whose variable $t$ is itself a function of $s$.

Another particular case of reduction pertains to the flows limited by an equilateral triangle which can be related to flows inside a particular rectangle, that is, as we shall see, to the elliptic functions and integrals.

In fact, we shall study the field of a doublet at one of the vertices of the triangle, corresponding to a function $t(u)$ defined by

$$\frac{du}{dt} = \left(1 - t^2\right)^{-\frac{2}{3}}$$

If we set

$$1 - t^2 = y^3$$

then

$$\frac{du}{dy} = -\frac{3}{2}\left(1 - y^3\right)^{-\frac{1}{2}}$$

and $y(u)$ defines, as the later chapters will show, an elliptic integral with a real modulus.

In general, numerous cases of reduction of the functions of Schwarz exist which may be found by analytical or geometrical methods. The subject of the present report is limited to the study of the reductions of the elliptic functions and integrals with real modulus which will form the subject of the following chapters.

## 1.7 Examples of Application

Assume that turning vanes for a right-angle elbow diffuser intended for a wind tunnel have to be defined.

One intends to construct the vanes very simply by means of one metal sheet reinforced with the aid of a second, soldered to the first (fig. 13). For the setup of the calculation, the thickness of the sheets is neglected and the regions of the trailing edge and the leading edge are assumed to be

straight. It is desired, moreover, to orient the leading edge so as to avoid its becoming bent.

Let $F(z)$ be the complex potential of the flows (fig. 14). To choose the distribution of velocities, it is convenient to construct the inverse hodograph

$$\zeta(f) = \frac{dz}{dF}$$

Let us follow the streamline which divides at the leading edge (figs. 14 and 15).

From B to C, the direction is constant and $\zeta$ varies on a radius OBC inclined by $\epsilon$.

From C to D, the direction of $\zeta$ changes and its intensity increases. Its extremity describes the curve CD.

From D to E, the direction is again constant and $\zeta$ describes the radius DE, inclined by $\frac{\pi}{2} + \epsilon'$.

The same reasoning applied to the upper surface of the vane fixes as limits of the field $F(\zeta)$ a curvilinear quadrilateral CDD'C'. The variation of the velocity will be quite continuous if logarithmic spirals are chosen for the curves CD and C'D'.

Let us now carry out the transformation $\ln\zeta$. The curvilinear quadrilateral is transformed into a parallelogram (fig. 16(a)), which itself corresponds to a circle or to a straight line by means of a Schwarz transformation (fig. 16(b))

$$\frac{d(\ln\zeta)}{d\chi} = C_o(\sin \chi)^{-\frac{1}{2}+\delta} \sin(\chi_1 - \chi)^{-\frac{1}{2}-\delta}$$

where the real part is chosen for $0 < \chi < \chi_1$ and where $C_o$ is a real constant determined by the condition that the imaginary variation of $\ln\zeta$ between C and D is

$$i\left(\frac{\pi}{2} + \epsilon + \epsilon'\right)$$

$$\frac{\pi}{2} + \epsilon + \epsilon' = C_o \cos \delta \int_{\chi_1}^{\pi} [\sin \chi]^{-\frac{1}{2}+\delta} [\sin(\chi_1 - \chi)]^{-\frac{1}{2}-\delta} d\chi$$

In order to choose the value of $X_A$ corresponding to the image of the point at infinity upstream, one will make $X$ vary by imaginary values, starting from an arbitrary real value $X_2$ such that $0 < X_2 < X_1$

$$i\epsilon = C_o\, I\int_{X_2}^{X_A} \left[\sin X\right]^{-\frac{1}{2}+\delta} \left[\sin(X_1 - X)\right]^{-\frac{1}{2}-\delta} dX$$

stopping the integration at the value $X_1 = X_2 + i\eta_2$ such that this equation is satisfied. One will operate in the same manner for defining $X_G$, corresponding to the image at infinity downstream, starting from a value $\pi + X_3$ such that $0 < X_3 < X_1$

$$i\epsilon' = C_o\, I\int_{\pi+X_3}^{X_G} \left[-\sin X\right]^{-\frac{1}{2}+\delta} \left[\sin(X - X_1)\right]^{-\frac{1}{2}-\delta} dX$$

Practically, for sufficiently small values of $\epsilon$ and $\epsilon'$

$$i\eta_2 \sim X_A - X_2 \sim i\epsilon\left[\sin X_3\right]^{\frac{1}{2}-\delta} \left[\sin(X_1 - X_2)\right]^{\frac{1}{2}+\delta}$$

$$i\eta_3 \sim X_3 - X_G \sim i\epsilon' \left[\sin X_3\right]^{\frac{1}{2}-\delta} \left[\sin(X_1 - X_3)\right]^{\frac{1}{2}+\delta}$$

It remains to define $F(X)$, the field of the initial potential in the plane of the variable $X$, or rather $F\left(e^{iX}\right)$ in the plane of the variable $e^{iX}$. This field is determined by its singularities which are source vortices in: $e^{iX_A}$ and $e^{iX_G}$ as well as in their images with respect to the unit circle in the plane of $e^{iX}$ (fig. 17).

The strength of a sink is equal to the strength of a source, and may be determined exactly by the condition that the imaginary part of $F$ varies by 1 from one vane to the next

$$2\pi F = (1 + i\tan\alpha)\ln\left(e^{iX} - e^{iX_2-\eta_2}\right) - (1 + i\tan\beta)\ln\left(e^{iX} + e^{iX_3-\eta_3}\right) +$$

$$(1 - i\tan\alpha)\ln\left(e^{iX} - e^{-iX_3+\eta_3}\right) - (1 - i\tan\beta)\ln\left(e^{iX} + e^{-iX_3+\eta_3}\right)$$

The constants $\tan\alpha$ and $\tan\beta$, put in this form for facilitating the ultimate notation, must be such that the complex variable $z$ in the plane of the physical variable will be uniform. For this it is necessary that the integral $\int \zeta \, dF$ taken on the contour CDD'C' of the hodograph be zero (fig. 15). Since the function $F$ presents singularities only at A and G inside the contour, the latter may be replaced by a loop enclosing A and G (fig. 18). One will note that

$$\zeta \, dF = \zeta \frac{dF}{dX} dX$$

where $\zeta$ and $dF/dX$ are uniform functions of $X$ inside the contour.

The portions of the integration on the two branches of the loop connecting A and G thus compensate one another, and it suffices to evaluate the residues at A and G.

In the neighborhood of A

$$\int \zeta \, dF \sim \zeta_A \int dF \sim i(1 + i \tan\alpha)\zeta_A$$

since the first logarithm of the expression of $F$ increases by $2\pi i$ for a contour enclosing A, and the others are uniform. Likewise, in the neighborhood of G

$$\int \zeta \, dF \sim \zeta_G \int dF \sim -i(1 + i \tan\beta)\zeta_G$$

The complex variable $z$ will be uniform if

$$i(1 + i \tan\alpha)\zeta_A - i(1 + i \tan\beta)\zeta_G = 0$$

$$e^{i(\alpha-\beta)} \frac{\cos\beta}{\cos\alpha} = \frac{\zeta_G}{\zeta_A}$$

For a deviation by a right angle, the argument of $\zeta_G/\zeta_A$ is

$$\alpha - \beta = \frac{\pi}{2}$$

and

$$\tan\alpha = -\cot\beta = \left|\frac{\zeta_G}{\zeta_A}\right|$$

The pitch of the vanes is the variation of $\int \zeta \, dF$ for a closed contour surrounding A or G. This variation has already been calculated above:

$$i(1 + i \tan \alpha)\zeta_A = ie^{i\alpha} \frac{\zeta_A}{\cos \alpha} = i\left(\zeta_A + \zeta_G\right)$$

The numerical calculation of the profile, for a real X, offers no difficulties. One must especially be careful to choose correctly the arbitrary initial values; the hodograph method prohibits fixing a priori the geometrical values in the physical plane. The angles $\epsilon$ and $\epsilon'$, for instance, determine indirectly the ratio between the chord of the profiles and the pitch of the vanes, for it is clear that with short profiles considerable velocity differences between infinity upstream and the leading edge, on one hand, between the trailing edge and infinity downstream, on the other, must be accepted. In compensation, the given conditions in the hodograph plane permit imposing on the velocity a quite continuous variation favorable to a sound flow of the real fluid.

There exist methods of electrical analogy more powerful and less laborious for designing lattices of blades.

The example above is intended especially for illustrating a study of analytic functions, performed in a manner suited to the needs of engineers.

## 2. ELLIPTIC INTEGRALS OF THE FIRST KIND AND ELLIPTIC FUNCTIONS

### 2.0 Definition

The elliptic integral of the first kind with real modulus is defined by

$$u = \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

The radical is positive for $x = 0$. The coefficient $k$ is real and smaller than one. Its square $k^2$ is the modulus. The inverse function is denoted by

$$x = \mathrm{sn}(u,k)$$

or, more simply, by $\mathrm{sn}\, u$ when $k$ does not vary in the course of the calculations. To this function are associated

$$x_1 = \text{cn } u = \sqrt{1 - \text{sn}^2 u}$$

$$x_2 = \text{dn } u = \sqrt{1 - k^2 \text{sn}^2 u}$$

where the radicals are positive for $u = 0$.

The inverse functions of cn u and dn u may be defined directly by integrals

$$u = -\int_1^{x_1} \frac{dx_1}{\sqrt{\left(1 - x_1^2\right)\left(1 - k^2 + k^2 x_1^2\right)}}$$

$$u = -\int_1^{x_2} \frac{dx_2}{\sqrt{\left(1 - x_2^2\right)\left(-1 + k^2 + x_2^2\right)}}$$

They could therefore be included in the definition of sn u if the restrictions fixed for the choice $k$ were lifted; the modulus is only under the restriction that it be real. It is more convenient for what follows to use the three functions for an equal value of $k$, smaller than one.

The two functions sn u and dn u are particular cases of the functions $t(u)$ defined in paragraph 1.4 and determine flows around doublets in a rectangle. The function cn u likewise defines a flow containing doublets, but with exchange of output between the doublets.

The three functions are represented by figure 19. The extension of the field of cn u by symmetry around a vertical side would complete the rectangle along which cn u is real but showing that this function is not, like sn u and dn u, one of the functions $t(u)$ defined in the preceding chapter for an arbitrary polygon. Its field contains two doublets on the perimeter of the rectangle.

The analytic continuations of the functions sn u, cn u, dn u, defined by symmetries around the sides of the rectangles are uniform and periodical. The same is true for any rational function of the three functions above which is called an elliptic function with real modulus. Such a function is not generally real on the contour of a rectangle and does no longer define a flow in a rectangle.

The derivatives of the functions sn u, cn u, dn u result immediately from their definitions

$$\frac{d}{du}(\mathrm{sn}\ u) = \mathrm{cn}\ u\ \mathrm{dn}\ u$$

$$\frac{d}{du}(\mathrm{cn}\ u) = -\mathrm{sn}\ u\ \mathrm{dn}\ u$$

$$\frac{d}{du}(\mathrm{dn}\ u) = -k^2 \mathrm{sn}\ u\ \mathrm{cn}\ u$$

In order to normalize the three functions as in the previous chapter in linking them to the field of the vortex at the center of the rectangle, it suffices to set

$$k = \sin \Theta$$

$$x = \mathrm{sn}\ u = \frac{\sin \chi}{\sin \Theta}$$

$$u = \int_0^{\chi} \frac{d\chi}{\sqrt{\sin^2\Theta - \sin^2\chi}} = \int_0^{\chi} \frac{d\chi}{2\sqrt{\sin\frac{\Theta - \chi}{2}\sin\frac{\Theta + \chi}{2}\sin\frac{\Theta + \pi + \chi}{2}\sin\frac{\Theta + \pi - \chi}{2}}}$$

The expressions of dn u and cn u as functions of $\chi$ are

$$\mathrm{dn}\ u = \cos \chi$$

$$\mathrm{cn}\ u = \frac{\sqrt{\sin^2\Theta - \sin^2\chi}}{\sin \Theta}$$

The derivative of $\chi(u)$ is

$$\frac{d\chi}{du} = \sqrt{\sin^2\Theta - \sin^2\chi} = \sin \Theta\ \mathrm{cn}\ u$$

The function $\chi(u)$ is periodic except for a multiple of $2\pi$ which depends on the chosen sections among the vortices. Since the direction of the vortices is reversed by symmetry, the periods are twice the sides of the defining rectangle of $\chi(u)$, for instance $4K$ and $4iK'$ where

$$K = \int_0^{\Theta} \frac{d\chi}{\sqrt{\sin^2\Theta - \sin^2\chi}}$$

$$iK' = \int_{\Theta}^{\pi/2} \frac{d\chi}{\sqrt{\sin^2\Theta - \sin^2\chi}} = i\int_0^{\pi/2-\Theta} \frac{d\left(\frac{\pi}{2} - \chi\right)}{\sqrt{\sin^2\left(\frac{\pi}{2} - \Theta\right) - \sin^2\left(\frac{\pi}{2} - \chi\right)}}$$

$$K'(\Theta) = K\left(\frac{\pi}{2} - \Theta\right)$$

The function $\chi(u)$ admits, moreover, the period $2K + 2iK'$ since two successive symmetries reestablish the direction of the vortices.

## 2.1 Logarithms of the Elliptic Functions

The logarithm of an elliptic function, multiplied by a complex coefficient, is represented by the doubly periodic field of source vortices and of sink vortices, corresponding to the poles and to the zeros of the elliptic function. The sum of such logarithms is of the same nature but, if the coefficients of the logarithms have no simple common measure, the intensities of the sources and the circulations of the vortices are varied.

The logarithms of $\mathrm{sn}\ u$, $\mathrm{cn}\ u$, $\mathrm{dn}\ u$ define the fields of figure 21

$$\ln \mathrm{sn}\ u = \ln \sin \chi - \ln \sin \Theta$$

$$\ln \mathrm{cn}\ u = \frac{1}{2} \ln \left[\sin(\Theta - \chi)\right] + \frac{1}{2} \ln \left[\sin(\Theta + \chi)\right] - \ln \sin \Theta$$

$$\ln \mathrm{dn}\ u = \ln \cos \chi$$

If the above fields are derived graphically, there corresponds to each source or sink a doublet, and to each straight line on which a function has a constant imaginary part while "du" is real or purely imaginary, corresponds a real or purely imaginary value of the derivative.

This operation furnishes the fields of figure 22

$$\frac{d}{du}\left[\ln \mathrm{sn}\ u\right] = \frac{\mathrm{cn}\ u\ \mathrm{dn}\ u}{\mathrm{sn}\ u}$$

$$\frac{d}{du}\left[\ln \mathrm{cn}\ u\right] = -\frac{\mathrm{sn}\ u\ \mathrm{dn}\ u}{\mathrm{cn}\ u}$$

$$\frac{d}{du}\left[\ln \mathrm{dn}\ u\right] = -\sin^2\Theta\ \frac{\mathrm{sn}\ u\ \mathrm{cn}\ u}{\mathrm{dn}\ u}$$

The last field is that of the function sn u, except for one constant and for one factor, for a ratio of the periods twice that of the original function. The conjugate field of the first, with, for instance, i cn u dn u/sn u, corresponds to the function dn u for a ratio of the periods half that of the original function, in addition, with a change of origin. The later calculations will furnish more convenient methods for multiplying or dividing the ratio of the periods.

If the field of the logarithm of an elliptic function is that of source vortices, the inverse does not hold true, as the later study will show. Even in the case where the representation of a field of source vortices is possible by the logarithm of an elliptic function, the finding of this function is difficult. The following example which will soon be useful shows the origin of the difficulty.

Let the function

$$F(u) = \ln \frac{\operatorname{cn} u - \operatorname{cn} u_1}{\operatorname{cn} u - \operatorname{cn} u_2}$$

be selected for representing a source of the strength $2\pi$ at $u = u_1$ feeding a sink, or a source of intensity $-2\pi$, at $u = u_2$.

The field of $F(u)$ contains actually sources of all values of $u$ for which $\operatorname{cn} u = \operatorname{cn} u_1$

$$u = u_1$$

$$u = -u_1$$

$$u = u_1 + 2K + 2iK'$$

$$u = -u_1 + 2K + 2iK'$$

and at all points deviating from the preceding ones by multiples of $4K$ and of $4iK'$. It is sometimes possible to eliminate the parasitic singularities introduced in order to conserve only those whose representation is desired.

A particular case is that of a field of source vortices inside a rectangle which has already been treated for an arbitrary polygon. The logarithm then allows for a constant imaginary part on the contour of the rectangle and the transformation $z = e^{i\chi(u)}$ leads, by analytic continuation to the exterior of the circle of radius 1 in the z-plane, to a field of source vortices defined in the entire plane.

$$F(u) = \Sigma A_n \ln\left[z(u) - z(u_n)\right]$$

where

$$z = \mathrm{dn}\, u + i \sin \Theta\, \mathrm{sn}\, u$$

If, in particular, the values of the coefficients $A_n$ are real numbers, the function $F$ is the logarithm of an elliptic function.

## 2.2 Change of Origin and Theorem of Addition

The problem of change of origin for an arbitrary elliptic function can be reduced to the problem of change of origin of the function $X(u)$, that is to say, to the definition of $X(u + v)$ as a function of $X(u)$, since the elliptic functions are expressed as functions of $X$.

The study may be carried out by conformal transformations, first for a real $v$, then for a purely imaginary $v$, and finally for a combination.

It is more direct and more convenient to define the function $X(u + v)$ by its singularities in the u-plane which are vortices with the circulation $\pm 2\pi$. The conjugate field $iX(u + v)$ is that of sources of intensities $\pm 2\pi$.

This is a case where it is possible to combine the singularities of elementary elliptic functions, and one finds

$$2iX(u + v) = \ln\left[\frac{\mathrm{cn}\, u - \mathrm{cn}(v - iK')}{\mathrm{cn}\, u - \mathrm{cn}(v + iK')}\,\frac{\mathrm{cn}(u + iK') - \mathrm{cn}(v + 2iK')}{\mathrm{cn}(u - iK') - \mathrm{cn}(v - 2iK')}\right] + C^{te}$$

The singularities of the second term are, in fact, partly those of sources of intensity $2\pi$ at the following points, defined except for multiples of $4K$ and of $4iK'$

$$\begin{array}{cccc} v - iK' & -v + iK' & v + iK' + 2K & -v - iK' + 2K \\ v + iK' & -v + iK' & v - iK' + 2K & -v - iK' + 2K \end{array}$$

and partly those of sinks of intensity $2\pi$ at

$$\begin{array}{cccc} v + iK' & -v - iK' & v - iK' + 2K & -v + iK' + 2K \\ v - iK' & -v - iK' & v + iK' + 2K & -v + iK' + 2K \end{array}$$

still except for multiples of $4K$ and of $4iK'$.

Some of these singularities compensate one another, whereas the effects of the others are additive. The field of the second term is definitely that of sources of intensity $4\pi$ at

$$-v + iK' \qquad\qquad -v - iK' + 2K$$

and of sinks of intensity $4\pi$ at

$$-v - iK' \qquad\qquad -v + iK' + 2K$$

which are, of course, the singularities of $2i\chi(u + v)$.

If $v$ and $u$ tend successively toward zero, the constant appears as zero.

It remains to link $\mathrm{cn}(u - iK') = -\mathrm{cn}(u + iK')$ to $\chi(u)$, that is, to treat the problem of change of origin initially posed in the particular case where $v = \pm iK'$.

The function $i\,\mathrm{cn}(u - iK')$ is real (fig. 24) on the defining rectangle of $\chi(u)$. It suffices therefore to define it in the plane of $z = e^{i\chi(u)}$ where it is represented by doublets at +1 and -1

$$i\,\mathrm{cn}(u - iK') = \frac{i}{z + 1} - \frac{i}{z - 1} + C_1$$

$$= -\cot \chi(u) + C_2$$

The constant $C_2$ may be specified by

$$u = K + iK' \quad \text{where} \quad \mathrm{cn}\,u = 0 \quad \text{and} \quad \chi(u) = \frac{\pi}{2}$$

It is zero

$$i\,\mathrm{cn}(u - iK') = -\cot \chi(u) = -\frac{\mathrm{dn}\,u}{\sin \Theta\ \mathrm{sn}\,u}$$

The formula of the change of origin assumes, consequently, the form

$$2i\chi(u + v) = \ln\left[\frac{\sin \Theta\ \mathrm{sn}\,v\ \mathrm{cn}\,u - i\,\mathrm{dn}\,v}{\sin \Theta\ \mathrm{sn}\,v\ \mathrm{cn}\,u + i\,\mathrm{dn}\,v} \times \frac{\sin \Theta\ \mathrm{sn}\,u\ \mathrm{cn}\,v - i\,\mathrm{dn}\,u}{\sin \Theta\ \mathrm{sn}\,u\ \mathrm{cn}\,v + i\,\mathrm{dn}\,u}\right]$$

$$\chi(u + v) = \arctan\left[\mathrm{cn}\,v \tan \chi(u)\right] + \arctan\left[\mathrm{cn}\,u \tan \chi(v)\right]$$

From this relationship result the classical addition formulas

$$\operatorname{sn}(u + v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - \sin^2\Theta \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{cn}(u + v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v}{1 - \sin^2\Theta \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{dn}(u + v) = \frac{\operatorname{dn} u \operatorname{dn} v - \sin^2\Theta \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v}{1 - \sin^2\Theta \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

where the denominator is the square of the common modulus of the four terms which appear in the expression of $2i\chi(u + v)$.

If one has set $v = u_1$ and $v + u = -u_2$, the functions $\chi(u)$, $\chi(u_1)$, $\chi(u_2)$ play symmetrical roles for

$$u + u_1 + u_2 = 0$$

It is easy to establish several symmetrical relations between the three functions $\chi$, but these relations have the disadvantage of not defining each of the functions from the two others in a uniform manner.

The addition formulas are valid for any arbitrary $u$ and $v$, but for a real $u$ and a purely imaginary $v$, they reduce the calculation of the $\chi$ function regarding a complex variable to the calculation of this function for the real values and the purely imaginary values. It is sufficient to change in the formulas $v$ to $-iv$.

More conveniently, one may note that the mode of definition of $\chi$ by a symmetrical vortex establishes the relation

$$\chi\left[K + iv, \sin \chi_\Theta\right] = \frac{\pi}{2} + \chi\left[v - K', \cos \Theta\right]$$

or simplifying the notation

$$\chi(K + iv) = \frac{\pi}{2} + \chi'(v - K')$$

$$\operatorname{sn}(K + iv) = \frac{\sin\left[\chi(K + iv)\right]}{\sin \Theta} = \frac{\cos\left[\chi'(v - K')\right]}{\sin \Theta} = \frac{\operatorname{dn}(v - K')}{\sin \Theta}$$

$$\mathrm{dn}(K + iv) = \cos\left[\chi(K + iv)\right] = -\sin\left[\chi'(v - K')\right] = -\cos\Theta\,\mathrm{sn}(v - K')$$

$$\mathrm{cn}(K + iv) = \sqrt{1 - \frac{\mathrm{sn}^2(K + iv)}{\sin^2\Theta}} = -i\cot\Theta\,\mathrm{cn}(v - K')$$

It is then indicated to replace in the addition formulas $u$ by $u - K$, and $v$ by $-K - iv$. The functions of $u + iv$ will thus be expressed with the aid of functions of $u - K$ and of $v - K'$; the addition formulas, in their normal forms, permit transforming these functions.

The calculation leads to the classical formulas

$$\chi(u + iv) = \arctan\left[\frac{\tan\chi(u)}{\mathrm{cn}'v}\right] + i\,\mathrm{arg\,tanh}\left[\mathrm{cn}\,u\tan\chi'(v)\tan\Theta\right]$$

$$\mathrm{sn}(u + iv) = \frac{\mathrm{sn}\,u\,\mathrm{dn}'\,v + i\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}'\,v\,\mathrm{cn}\,v}{\mathrm{cn}'^2 v + \sin^2\Theta\,\mathrm{sn}^2 u\,\mathrm{sn}'^2 v}$$

$$\mathrm{cn}(u + iv) = \frac{\mathrm{cn}\,u\,\mathrm{cn}'\,v - i\,\mathrm{sn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}'\,v\,\mathrm{dn}'v}{\mathrm{cn}'^2 v + \sin^2\Theta\,\mathrm{sn}^2 u\,\mathrm{sn}'^2 v}$$

$$\mathrm{dn}(u + iv) = \frac{\mathrm{dn}\,u\,\mathrm{cn}'\,v\,\mathrm{dn}'\,v - i\sin^2\Theta\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{sn}'\,v}{\mathrm{cn}'^2 v + \sin^2\Theta\,\mathrm{sn}^2 u\,\mathrm{sn}'^2 v}$$

Thus, it is sufficient to calculate $\chi(u)$ from which the other functions can be deduced, for the values of $\Theta$ between 0 and $\pi/2$, and the real values of $u$ between 0 and $K$. However, the addition formula establishes, moreover, for a change of origin by a fourth of the real period

$$\mathrm{dn}(u - K) = \mathrm{dn}(K - u) = \frac{\cos\Theta}{\mathrm{dn}\,u}$$

$$\cos\left[\chi(K - u)\right]\cos\left[\chi(u)\right] = \cos\Theta$$

It suffices therefore to make $u$ vary from 0 to $K/2$ where

$$\mathrm{dn}\,\frac{K}{2} = \sqrt{\cos\Theta} \qquad \mathrm{sn}\,\frac{K}{2} = \frac{1}{\sqrt{2}\cos\frac{\Theta}{2}} \qquad \mathrm{cn}\,\frac{K}{2} = \frac{\sqrt{\cos\Theta}}{\sqrt{2}\cos\frac{\Theta}{2}}$$

We remark furthermore that

$$\operatorname{cn} u = \frac{\tan\left[X(K - u)\right]}{\tan \Theta}$$

and this formula permits calculation of $\operatorname{cn} u$ from $X(K - u)$, known from 0 to $K$, without use of a radical.

The addition formula of $X$ may be written

$$X(u + v) = \operatorname{arc} \tan\left[\frac{\tan X(u) \tan X(K - v)}{\tan \Theta}\right] + \operatorname{arc} \tan\left[\frac{\tan X(v) \tan X(K - u)}{\tan \Theta}\right]$$

This expression explains why use of the symmetrical relationships between the three functions $X(u)$, $X(-v)$, and $X(-u - v)$ is not very convenient. The function $X(u + v)$ is actually expressed with the aid of four nonindependent functions.

One could set up a parallel argument regarding a change of origin by a fourth of the imaginary period

$$\sin X(u + iK') \sin X(u) = \sin \Theta$$

Another classical means for calculating $\operatorname{cn} u$ without using a radical consists in setting

$$\operatorname{cn} u = \cos \varphi = - \frac{\sinh \psi}{\tan \Theta}$$

$$\operatorname{sn} u = \sin \varphi = \frac{\sin X}{\sin \Theta}$$

$$\operatorname{dn} u = \cos \Theta \cosh \psi = \cos X$$

$$2i\varphi = \ln \frac{\operatorname{cn} u + i \operatorname{sn} u}{\operatorname{cn} u - i \operatorname{sn} u}$$

$$2\psi = \ln \frac{\sin \Theta \operatorname{dn} u + \operatorname{cn} u}{\sin \Theta \operatorname{dn} u - \operatorname{cn} u}$$

The fields of the functions $\varphi(u)$ and $\psi(u)$ are represented by figure 25. The functions $i\varphi(u)$ and $\psi(u)$ have fields which present the same singularities as $\ln\left[\operatorname{sn} u\right]$ and $\ln\left[\operatorname{dn} u\right]$ except for the ratio of the periods and the origin. This could furnish a second means for doubling or dividing the ratio of the periods by two.

The derivatives of the functions $\varphi$ and $\psi$ are

$$\frac{d\varphi}{du} = \mathrm{dn}\ u$$

$$\frac{d\psi}{du} = -\sin\Theta\ \mathrm{sn}\ u$$

The functions $\psi$, $\chi$, $\varphi$ therefore integrate, except for one factor, the functions sn u, cn u, dn u.

Changes of origin permit integration of other simple elliptic functions with the functions $\chi$, $\varphi$, $\psi$ calculated for $u + K$; $u + iK'$; $u + iK + iK'$.

### 2.3 Gauss and Landen Transformations

The most direct and geometrically most significant method for studying the doubling of the ratio of the periods consists in the study of the $\chi(u)$ function.

Let us consider two functions $\chi(u,k)$ and $\chi_1\left(u_1,k_1\right)$ of the same origin (fig. 26).

The scale of $u_1$ is chosen in such a manner that the real periods of the two functions are the same in the u-plane

$$\frac{u_1}{K_1} = \frac{u}{K}$$

The ratios of the periods are

$$\frac{iK'_1}{K_1} = \frac{i}{2}\frac{K'}{K}$$

In the plane of the variable $z = e^{i\chi}$, the function $\chi_1$ is represented by two vortices at $z_0 = e^{i\chi\left(\frac{iK'}{2}\right)}$ and $-z_0$ inside the unit circle the circulations of which are $\pm 2\pi$. It is defined by these singularities and their images with respect to the circle.

$$\chi_1 = i\ \ln\left[-\frac{z + z_0}{z - z_0}\frac{z - \frac{1}{z_0}}{z + \frac{1}{z_0}}\right] + C^{te}$$

The minus sign was introduced into the logarithm so that the constant should become zero when $\chi_1$ is calculated for $z = 1$ where $u_1 = u = 0$ .

$$\chi_1 = 2 \text{ arc tan}\left[\frac{2z_o \sin \chi}{1 - z_o^2}\right]$$

At $u = K + iK'$, the function $\chi$ is $\pi/2$ and the function $\chi_1$ is $\pi - \Theta_1$

$$\pi - \Theta_1 = 2 \text{ arc tan}\left[\frac{2z_o}{1 - z_o^2}\right]$$

$$\cot \frac{\Theta_1}{2} = \frac{i}{\sin\left[i\chi\left(\frac{iK'}{2}\right)\right]}$$

We remark that

$$\sin\left[i\chi\left(\frac{iK'}{2}\right)\right] = \sin \Theta \text{ sn } \frac{iK'}{2} = i \sin \Theta \frac{\text{sn}' \frac{K'}{2}}{\text{cn}' \frac{K'}{2}} = i \sqrt{\sin \Theta}$$

and it appears that $\Theta_1$ is connected with $\Theta$ by

$$\tan^2 \frac{\Theta_1}{2} = \sin \Theta$$

whereas the relation between $\chi_1$ and $\chi$ is

$$\tan \frac{\Theta_1}{2} \tan \frac{\chi_1}{2} = \sin \chi$$

Deriving this last relationship and calculating the derivatives for $\chi_1 = \chi = 0$, one establishes that

$$\frac{K_1}{K} = 2i \frac{K_1'}{K_1} = \frac{1}{\cos^2 \frac{\Theta_1}{2}}$$

and the transformation is finally

$$\tan \frac{\Theta_1}{2} \tan\left[\frac{\chi\left(u_1, \sin \Theta_1\right)}{2}\right] = \sin\left[\chi\left(u_1 \cos^2 \frac{\Theta_1}{2}, \tan^2 \frac{\Theta_1}{2}\right)\right]$$

It is possible to deduce from this the relations of Gauss

$$\text{sn}\left(u_1, \sin \Theta_1\right) = \frac{\text{sn}\left[u_1 \cos^2 \frac{\Theta_1}{2}, \tan^2 \frac{\Theta_1}{2}\right]}{\cos^2 \frac{\Theta_1}{2} + \sin^2 \frac{\Theta_1}{2} \text{ sn}^2\left[u_1 \cos^2 \frac{\Theta_1}{2}, \tan^2 \frac{\Theta_1}{2}\right]}$$

$$\operatorname{cn}\left(u_1, \sin\Theta_1\right) = \frac{\cos^2\frac{\Theta_1}{2}\operatorname{cn}\left[u_1\cos^2\frac{\Theta_1}{2}, \tan^2\frac{\Theta_1}{2}\right]\operatorname{dn}\left[u_1\cos^2\frac{\Theta_1}{2}, \tan^2\frac{\Theta_1}{2}\right]}{\cos^2\frac{\Theta_1}{2} + \sin^2\frac{\Theta_1}{2}\operatorname{sn}^2\left[u_1\cos^2\frac{\Theta_1}{2}, \tan^2\frac{\Theta_1}{2}\right]}$$

$$\operatorname{dn}\left(u_1, \sin\Theta_1\right) = \frac{\cos^2\frac{\Theta_1}{2} + \sin^2\frac{\Theta_1}{2}\operatorname{sn}^2\left[u_1\cos^2\frac{\Theta_1}{2}, \tan^2\frac{\Theta_1}{2}\right]}{\cos^2\frac{\Theta_1}{2} - \sin^2\frac{\Theta_1}{2}\operatorname{sn}^2\left[u_1\cos^2\frac{\Theta_1}{2}, \tan^2\frac{\Theta_1}{2}\right]}$$

the use of which is less convenient than that of the trigonometric relation between the $X$ functions.

Making use of the Gauss transformation, one can obtain from it many others by change of origin and interchange of the axes with the aid of the addition formulas. The most frequently used one is the transformation of Landen which defines the function $X_2\left(u_2,k_2\right)$ as a function of $X(u,k)$ for the same origin but a ratio of the real period doubled (fig. 27).

It is clear that the Gauss transformation applied around the point $K_2 + iK'_2$ and for the axes turned by a right angle requires a function represented by $X(u)$ to correspond to $X_2$ except for a displacement from the origin by $-K/2$. This geometric consideration guides the calculations.

First of all, the symmetry of definition of $X(u)$ permits making a first change of origin

$$X_2\left(u_2, \sin\Theta_2\right) = \frac{\pi}{2} + X\left[-i\left(u_2 - K_2 - iK'_2\right), \sin\left(\frac{\pi}{2} - \Theta_2\right)\right]$$

The transformation of Gauss defines then the vortex in the rectangle of twice the length-width ratio

$$\tan\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)\tan\left[\frac{X\left[-iu_2 + iK_2 - K'_2, \sin\left(\frac{\pi}{2} - \Theta_2\right)\right]}{2}\right] =$$

$$\sin\left[X\left(-iu_2 + iK_2 - K'_2\right)\cos^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right), \tan^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)\right]$$

Finally, a second change of origin leads to the function $X(u, \sin\Theta)$.

$$\chi\left[\left(-iu_2 + iK_2 - K'_2\right)\cos^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right),\ \tan^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)\right] = \chi\left[u + \frac{K}{2},\ \sin\Theta\right] - \frac{\pi}{2} =$$

$$\chi\left[\left(u - K_2 - iK'_2\right)\cos^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right) + K + iK',\ \tan^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)\right] - \frac{\pi}{2}$$

Setting the two last expressions equal:

$$\cos\Theta = \tan^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right) = \frac{1 - \sin\Theta_2}{1 + \sin\Theta_2}$$

$$\frac{2K_2}{K} = \frac{iK'_2}{iK'} = \frac{1}{\cos^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)}$$

$$u_2 = \frac{u}{\cos^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)}$$

It remains only to make the calculations with the aid of the addition formulas of $\chi(u)$

$$\tan\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)\tan\left[\frac{\chi\left(u_2,\ \sin\Theta_2\right)}{2} - \frac{\pi}{4}\right] = \sin\left[\chi\left(u + \frac{K}{2},\ \sin\Theta\right) - \frac{\pi}{2}\right]$$

$$= -\mathrm{dn}\left[u + \frac{K}{2},\ \sin\Theta\right]$$

Noting that

$$\mathrm{dn}\,\frac{K}{2} = \sqrt{\cos\Theta} = \tan\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right) \qquad \mathrm{sn}\,\frac{K}{2} = \cos\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right) \qquad \mathrm{cn}\,\frac{K}{2} = \sin\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)$$

we find that the addition formula of $\mathrm{dn}\,u$ furnishes

$$\tan\left[\frac{\pi}{4} - \frac{\chi\left(u_2,\ \sin\Theta_2\right)}{2}\right] = \frac{\mathrm{dn}\,u - (1 - \cos\Theta)\mathrm{sn}\,u\ \mathrm{cn}\,u}{1 - (1 - \cos\Theta)\mathrm{sn}^2u}$$

The notation of this complicated formula has been simplified. It is possible to deduce from it Landen's formulas

$$\mathrm{sn}\left(u_2,\ \sin\Theta_2\right) = \frac{1 - \tan^2\left(\frac{\pi}{4} - \frac{\chi_2}{2}\right)}{1 + \tan^2\left(\frac{\pi}{4} - \frac{\chi_2}{2}\right)}$$

$$\mathrm{dn}\left(u_2, \sin \Theta_2\right) = \frac{2 \tan\left(\frac{\pi}{4} - \frac{X_2}{2}\right)}{1 + \tan^2\left(\frac{\pi}{4} - \frac{X_2}{2}\right)}$$

$$\mathrm{sn}\left(u_2, \sin \Theta_2\right) = \left[1 - \tan^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)\right] \frac{\mathrm{sn}\ u\ \mathrm{cn}\ u}{\mathrm{dn}\ u}$$

$$\mathrm{cn}\left(u_2, \sin \Theta_2\right) = \frac{1 - \left[1 + \tan^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)\right] \mathrm{sn}^2 u}{\mathrm{dn}\ u}$$

$$\mathrm{dn}\left(u_2, \sin \Theta_2\right) = \frac{1 - \left[1 - \tan^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)\right] \mathrm{sn}^2 u}{\mathrm{dn}\ u}$$

where the elliptic functions $\mathrm{sn}\ u$, $\mathrm{cn}\ u$, $\mathrm{dn}\ u$, written in abbreviated form, represent

$$\mathrm{sn}\ u = \mathrm{sn}(u, \sin \Theta) = \mathrm{sn}\left[u_2 \cos^2\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right), \sqrt{1 - \tan^4\left(\frac{\pi}{4} - \frac{\Theta_2}{2}\right)}\right]$$

and so on.

The transformation of Landen permits approximating the logarithmic derivative of $\mathrm{dn}\ u$ to the function $\mathrm{sn}\ u$ for a doubled ratio of the periods.

$$\frac{d}{du}\left[\ln \mathrm{dn}(u, \sin \Theta)\right] = -(1 + \cos \Theta)\mathrm{sn}\left[u(1 + \cos \Theta), \frac{1 - \sin \Theta}{1 + \sin \Theta}\right]$$

The logarithmic derivative of $\mathrm{sn}\ u$ corresponds to a third type of transformation derived from that of Gauss by a change of origin, with imaginary complex variable, without interchange of the axes.

Likewise, the functions $e^{i\varphi(u)}$ and $e^{\psi(u)}$ have fields which can be linked to those of the elliptic functions by means of the ratio of the periods double or half.

## 2.4 Multiplication or Division of the Ratio of the Periods by an Odd Integer

Due to its analytic continuation by symmetries, the function $X(u)$ is represented by a field of vortices of alternate directions. If a

vortex on p is retained, in the sense of one of the periods, with p being an odd integer, the field established corresponds again to vortices of alternate directions, that is, to a function $\chi(u)$ for a ratio of the periods p times as large. The initial function $\chi(u)$ therefore is the sum of p functions $\chi(u)$ for ratios of periods p times as large and being deduced one from another by a displacement of the origin. This property does not exist if p is an even number as in the case of the transformations of Gauss and Landen.

Let us study first the functions $\chi(u)$ with the same origin for a modification of the imaginary periods (fig. 28), comparable to those of the Gauss transformation.

The function $\chi\left(u_1,k_1\right)$ for which the ratio between the imaginary period and the real period is p times as small as the corresponding ratio for the function $\chi(u,k)$ is defined by its singularities.

$$\chi\left(u_1,k_1\right) = \sum_{q=-\frac{p-1}{2}}^{q=\frac{p-1}{2}} (-1)^q \chi\left[u + \frac{iqK'}{p}, k\right] + C^{te}$$

Using the addition formula of $\chi(u)$ and noting that

$$\chi\left(i\frac{qK'}{p}, k\right) = -\chi\left(-\frac{iqK'}{p}, k\right)$$

one has, since $\chi$ is odd

$$\chi\left(u_1,k_1\right) = \chi(u,k) + 2\sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \text{ arc tan}\left[\frac{\tan \chi(u,k)}{\text{cn}\left(\frac{qK'}{p}, k'\right)}\right] + C^{te}$$

The constant is zero because, for a convenient determination of the arc tangents, all terms are zero for $u_1 = u = 0$ at the origin.

The values of $u_1$ and of $u$ are linked, according to figure 28

$$\frac{u_1}{K_1} = \frac{u}{K}$$

The constants $k_1 = \sin \theta_1$ and $k_2 = \sin \theta_2$ are likewise connected. It suffices to write for $u = K$ the relation between $\chi_1$ and $\chi$

$$\Theta_1 = \Theta + 2\sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \text{ arc tan}\left[\frac{\tan\Theta}{\text{cn}\left(\frac{qK'}{p}, \cos\Theta\right)}\right]$$

Finally, a differentiation of $\chi_1$ with respect to $u$ furnishes

$$\frac{\sin\Theta_1 \text{ cn}\left(u_1, \sin\Theta_1\right)}{\sin\Theta \text{ cn}(u, \sin\Theta)} \frac{K_1}{K} = 1+2\sum_{q=1}^{q=\frac{p-1}{2}} \frac{\left[1+\tan^2\chi(u, \sin\Theta)\right]\text{cn}\left[\frac{qK'}{p}, \cos\Theta\right]}{\text{cn}^2\left[\frac{qK'}{p}, \cos\Theta\right] + \tan^2\chi(u, \sin\Theta)}$$

The value of $K_1/K$ may be specified for $u_1 = u = 0$

$$\frac{K_1}{K} = p\frac{iK_1'}{iK_1} = \frac{\sin\Theta}{\sin\Theta_1}\left[1 + 2\sum_{q=1}^{q=\frac{p-1}{2}} \frac{(-1)^q}{\text{cn}\left(\frac{qK'}{p}, \cos\Theta\right)}\right]$$

Let us study now the multiplication of the real period by an odd integer $p$ (fig. 29) for functions $\chi\left(u_2,k_2\right)$ and $\chi(u,k)$ of the same origin.

The same argument leads to writing

$$\chi\left(u_2,k_2\right) = \sum_{q=-\frac{p-1}{2}}^{q=\frac{p-1}{2}} (-1)^q \chi\left(u + \frac{qK}{p}, k\right) + c^{te}$$

That is to say

$$\chi\left(u_2,k_2\right) = \chi(u,k) + 2\sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \text{ arc tan}\left[\tan\chi(u,k)\text{cn}\left(\frac{qK}{p}, k\right)\right]$$

The relation between $k_2 = \sin\Theta_2$ and $k = \sin\Theta$ will also be specified for $u = K$ and $u_2 = pK_2$

$$-\Theta_2 = \Theta + 2\sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \text{ arc tan}\left[\tan\Theta \text{ cn}\left(\frac{qK}{p}, \sin\Theta\right)\right]$$

and the relation between $K_2$ and $K$ will result also from the calculation of the derivatives for $u = u_2 = 0$

$$\frac{\sin \Theta_2 \operatorname{cn}\left(u_2, \sin \Theta_2\right)}{\sin \Theta \operatorname{cn}(u, \sin \Theta)} \frac{K_2}{K} = 1 + 2 \sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \frac{\left(1 + \tan^2\chi\right)\operatorname{cn}\left(\frac{qK}{p}, \sin \Theta\right)}{1 + \tan^2\chi \operatorname{cn}^2\left(\frac{qK}{p}, \sin \Theta\right)}$$

$$\frac{K_2}{K} = \frac{1}{p} \frac{iK'_2}{iK'} = \frac{\sin \Theta}{\sin \Theta_2}\left[1 + 2 \sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \operatorname{cn}\left(\frac{qK}{p}, \sin \Theta\right)\right]$$

The above transformations, combined with those of Gauss and Landen, permit linking the functions $\chi(u)$ for the ratios of the periods which have an arbitrary fraction as the quotient. However, the calculations are very complicated, and the transformations are especially useful for the approximate calculation of the elliptic functions.

## 2.5 The Functions $\varphi(u)$ and $\psi(u)$

The functions $\varphi(u)$ and $\psi(u)$ correspond, like $\chi(u)$, to alternating vortices or sources and give reason for use of the same method of addition of the singularities.

The function $\varphi(u)$ is defined by

$$\sin \varphi(u) = \operatorname{sn} u \qquad \cos \varphi(u) = \operatorname{cn} u$$

Its addition formula may be deduced from those of the elliptic functions.

We simplify the notation by designating by the subscript 1 the functions of the variable $u$, by 2 those of the variable $v$, and by 3 those of the variable $u + v$

$$\sin \varphi_3 = \operatorname{sn}_3 = \frac{\operatorname{sn}_1 \operatorname{cn}_2 \operatorname{dn}_2 + \operatorname{sn}_2 \operatorname{cn}_1 \operatorname{dn}_1}{1 - \sin^2\Theta \operatorname{sn}_1^2 \operatorname{sn}_2^2}$$

$$\cos \varphi_3 = \operatorname{cn}_3 = \frac{\operatorname{cn}_1 \operatorname{cn}_2 - \operatorname{sn}_1 \operatorname{dn}_1 \operatorname{sn}_2 \operatorname{dn}_2}{1 - \sin^2\Theta \operatorname{sn}_1^2 \operatorname{sn}_2^2}$$

$$e^{2i\varphi_3} = \frac{\cos\varphi_3 + i\sin\varphi_3}{\cos\varphi_3 - i\sin\varphi_3} = \frac{\left(\mathrm{cn}_1 + i\,\mathrm{sn}_1\,\mathrm{dn}_2\right)\left(\mathrm{cn}_2 + i\,\mathrm{sn}_2\,\mathrm{dn}_1\right)}{\left(\mathrm{cn}_1 - i\,\mathrm{sn}_1\,\mathrm{dn}_2\right)\left(\mathrm{cn}_2 - i\,\mathrm{sn}_2\,\mathrm{dn}_1\right)}$$

$$\varphi_3 = \arctan\left[\frac{\mathrm{sn}_1}{\mathrm{cn}_1}\,\mathrm{dn}_2\right] + \arctan\left[\frac{\mathrm{sn}_2}{\mathrm{cn}_2}\,\mathrm{dn}_1\right]$$

$$= \arctan\left[\mathrm{dn}_2 \tan\varphi_1\right] + \arctan\left[\mathrm{dn}_1 \tan\varphi_2\right]$$

For completing the symmetry with the formula relating to $X(u)$, one may note that

$$\mathrm{dn}\,u = i\tan\left[\varphi(K + iK' - u)\right]$$

The same calculation performed for the function $\psi(u)$ defined by

$$\cosh\psi = \frac{\mathrm{dn}\,u}{\cos\Theta} \qquad \sinh\psi = \tan\Theta\,\mathrm{cn}\,u$$

leads to the addition formula

$$\psi_3 = \arg\tanh\left[\sin\Theta\,\frac{\mathrm{cn}_1}{\mathrm{dn}_1}\,\frac{\mathrm{cn}_2}{\mathrm{dn}_2}\right] - \arg\tanh\left[\sin\Theta\,\mathrm{sn}_1\,\mathrm{sn}_2\right]$$

$$= \arg\tanh\left[\frac{\tanh\psi_1\,\tanh\psi_2}{\sin\Theta}\right] - \arg\tanh\left[\frac{\tanh\psi_1(K - u)\tanh\psi_2(K - v)}{\sin\Theta}\right]$$

The method of superposition for multiplication or division of the ratio of the periods by an odd integer may be applied to the functions $\varphi(u)$ and $\psi(u)$

$$\varphi\left(u_1,k_1\right) = \sum_{q=-\frac{p-1}{2}}^{q=\frac{p-1}{2}} (-1)^q\,\varphi\left(u + \frac{iqK'}{p},\ k\right)$$

$$= \varphi(u,k) + 2\sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \arctan\left[\frac{\tan\varphi(u,k)\,\mathrm{dn}\left(\frac{qK'}{p},\ k'\right)}{\mathrm{cn}\left(\frac{qK'}{p},\ k'\right)}\right]$$

$$\varphi\left(u_2,k_2\right) = \sum_{q=-\frac{p-1}{2}}^{q=\frac{p-1}{2}} (-1)^q \varphi\left(u + \frac{qK}{p}, k\right)$$

$$= \varphi(u,k) + 2 \sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \text{ arc tan}\left[\tan \varphi(u,k)\mathrm{dn}\left(\frac{qK}{p},k\right)\right]$$

For the function $\psi(u)$, one must be cautious because this function is not zero for $u = 0$, and in order to avoid introduction of a constant, it is convenient to calculate $\psi(K - u)$ which is zero for $u = 0$.

$$\psi\left(K - u_1,k_1\right) = \sum_{q=-\frac{p-1}{2}}^{q=\frac{p-1}{2}} (-1)^q \psi\left(K - u + \frac{iqK'}{p},k\right)$$

$$= \psi(K - u,k) + 2 \sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \text{ arg tanh}\left[\frac{\tanh \psi(K - u,k)}{\mathrm{dn}\left(\frac{qK'}{p},k'\right)}\right]$$

$$\psi\left(K - u_2,k_2\right) = \sum_{q=-\frac{p-1}{2}}^{q=\frac{p-1}{2}} (-1)^q \psi\left(K - u + \frac{qK}{p},k\right)$$

$$= \psi(K - u,k) + 2 \sum_{q=1}^{q=\frac{p-1}{2}} (-1)^q \text{ arg tanh}\left[\frac{\tanh \psi(K - u,k)\mathrm{cn}\left(\frac{qK}{p},k\right)}{\mathrm{dn}\left(\frac{qK}{p},k\right)}\right]$$

## 2.6 Expansions in Trigonometrical Series

Since the elliptic functions and their logarithms, the latter except for a multiple of $2i\pi$, are doubly periodic functions of $u$, it is possible to eliminate one of the periods by a logarithmic transformation, foregoing the symmetry of the roles of the two periods.

We choose to eliminate the period $4K$ by the transformation

$$2\pi i \frac{u}{4K} = \ln t \qquad t = e^{\frac{i\pi u}{2K}}$$

When $u$ increases by $4K$, the variable $t$ reassumes the same value as the elliptic function $F(u)$ which is a uniform function of $t$. When, in contrast, $u$ increases by $4iK'$, the variable $t$ is multiplied by $q^2 = e^{-2\pi\frac{K'}{K}}$ and thus is not a uniform function of $F$ since to a single value of $F$ there corresponds an infinite number of values of $t$ in geometrical progression with ratio $q^2$.

To the rectangle of the periods of the function $\chi(u)$, there corresponds in the $t$ plane a ring-shaped area bounded by the circle of the radius 1 and the circle of the radius $q^2$ (fig. 30) in which the field of the $\chi$ function is that of four vortices with the circulations $\pm 2\pi$ in

$$e^{\frac{i\pi(iK')}{2K}} = q^{\frac{1}{2}} \quad \text{in} \quad -q^{\frac{1}{2}} \quad \text{in} \quad q^{\frac{3}{2}} \quad \text{and in} \quad -q^{+\frac{3}{2}}$$

The neighboring rectangles are transformed into rings reduced or increased in the ratio $q^2$, and finally the $\chi$ function is defined in the entire plane by its singularities which are those of alternating vortices on the real axis of $t$ with real complex variables $\pm q^{\left(r+\frac{1}{2}\right)}$ where $r$ assumes all integer values, positive or negative.

$$i\chi = \sum_{r=-\infty}^{r=\infty} (-1)^r \ln \frac{t - q^{r+\frac{1}{2}}}{t + q^{r+\frac{1}{2}}} + C^{te}$$

It will be convenient for what follows to distinguish the positive and the negative values of $r$ by writing

$$i\chi = \sum_{0}^{\infty} (-1)^r \ln \frac{t - q^{r+\frac{1}{2}}}{t + q^{r+\frac{1}{2}}} \, \frac{q^{-r-\frac{1}{2}} + t}{q^{-r-\frac{1}{2}} - t} + C^{te}$$

The constant is zero as it appears when $u = X$ and $t = 1$.

Coming back to the variable $u$

$$i\chi = \sum_{0}^{\infty} (-1)^r \ln \left[\frac{\left(1 - q^{r+\frac{1}{2}} e^{-i\frac{\pi u}{2K}}\right)\left(1 + q^{r+\frac{1}{2}} e^{i\frac{\pi u}{2K}}\right)}{\left(1 + q^{r+\frac{1}{2}} e^{-i\frac{\pi u}{2K}}\right)\left(1 - q^{r+\frac{1}{2}} e^{i\frac{\pi u}{2K}}\right)}\right]$$

Since the parameter $q$ is essentially smaller than 1, every logarithm can be expanded in series, for the real values of $u$. The validity of the addition of the series is less evident but it is accepted without justification

$$i\chi(u) = 2 \sum_{s=0}^{s=\infty} \sum_{r=0}^{r=\infty} (-1)^r \frac{q^{\left(r+\frac{1}{2}\right)(2s+1)}}{2s+1}\left[e^{(2s+1)i\frac{\pi u}{2K}} - e^{-(2s+1)i\frac{\pi u}{2K}}\right]$$

$$\chi(u) = 4 \sum_{s=0}^{s=\infty} \frac{q^{s+\frac{1}{2}}}{1+q^{2s+1}} \frac{1}{2s+1} \sin\left[(2s+1)\frac{\pi u}{2K}\right]$$

The function $\chi(u)$ is thus represented by a trigonometric series, the coefficients of which are functions of $q$, which is itself proportional to the logarithm of the ratio of the periods. The corresponding value of $\Theta$ is $\chi(K)$

$$\Theta = 4 \sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} \frac{q^{s+\frac{1}{2}}}{1+q^{2s+1}}$$

and this alternating series is rapidly convergent.

For $K' = K$, in particular, the value of $q$ is $e^{-\pi}$ and that of $\Theta$ is $\pi/4$ whence the relation

$$\frac{\pi}{4} = \sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} \frac{e^{-\left(s+\frac{1}{2}\right)\pi}}{1+e^{-(2s+1)\pi}}$$

The expansion in trigonometric series of $\operatorname{cn} u$ is obtained by differentiation

$$\operatorname{cn} u = \frac{1}{\sin\Theta}\frac{d\chi}{du} = \frac{2\pi}{K\sin\Theta} \sum_{s=0}^{\infty} \frac{q^{s+\frac{1}{2}}}{1+q^{2s+1}} \cos\left[(2s+1)\frac{\pi u}{2K}\right]$$

In particular, the value of $K$ is defined for $u = 0$.

$$K = \frac{2\pi}{\sin \Theta} \sum_{s=0}^{\infty} \frac{q^{s+\frac{1}{2}}}{1 + q^{2s+1}}$$

$$= \frac{2\pi}{\sin \Theta} \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r \, q^{2\left(s+\frac{1}{2}\right)\left(r+\frac{1}{2}\right)}$$

$$= \frac{2\pi}{\sin \Theta} \left[ \sum_{n=0}^{\infty} q^{\left(\frac{2n+1}{2}\right)^2} \right]^2$$

The above series permit calculation of the elliptic functions much more easily than by numerical integration of the integrals of the first kind.

It is equally possible to deduce from the first expansion of $X$

$$X = 2 \sum_{r=0}^{\infty} (-1)^r \left[ \text{arc tan} \frac{q^{r+\frac{1}{2}} \sin \frac{\pi u}{2K}}{1 + q^{2r+\frac{1}{2}} \cos \frac{\pi u}{2K}} + \text{arc tan} \frac{q^{r+\frac{1}{2}} \sin \frac{\pi u}{2K}}{1 - q^{r+\frac{1}{2}} \cos \frac{\pi u}{2K}} \right]$$

$$= 2 \sum_{r=0}^{\infty} (-1)^r \, \text{arc tan} \left[ \frac{2q^{r+\frac{1}{2}} \sin \frac{\pi u}{2K}}{1 - q^{2r+1}} \right]$$

with, in particular,

$$\Theta = 2 \sum_{r=0}^{\infty} (-1)^r \, \text{arc tan} \left[ \frac{2q^{r+\frac{1}{2}}}{1 - q^{2r+1}} \right]$$

The functions $\varphi(u)$ and $\psi(u)$ give a basis for comparable calculations the principal results of which are

$$\varphi = \frac{\pi u}{2K} + 2 \sum_{s=1}^{\infty} \frac{q^s}{1 + q^{2s}} \frac{1}{s} \sin \left( s \frac{\pi u}{K} \right)$$

$$\text{dn } u = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{s=1}^{\infty} \frac{q^s}{1 + q^{2s}} \cos \left( s \frac{\pi u}{K} \right)$$

$$\psi = -2 \sum_{s=0}^{\infty} \frac{q^{s+\frac{1}{2}}}{1 - q^{2s+1}} \frac{1}{2s + 1} \cos\left[(2s + 1)\frac{\pi u}{2K}\right]$$

$$\operatorname{sn} u = \frac{2\pi}{K \sin \Theta} \sum_{s=0}^{\infty} \frac{q^{s+\frac{1}{2}}}{1 - q^{2s+1}} \sin\left[(2s + 1)\frac{\pi u}{2K}\right]$$

The value of $\operatorname{dn} u$ for $u = 0$ furnishes an expression of $K$ as a function of $q$ independent of $\Theta$

$$K = \frac{\pi}{2} + 2\pi \sum_{s=1}^{\infty} \frac{q^s}{1 + q^{2s}}$$

$$= \frac{\pi}{2} + 2\pi \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} (-1)^{r-1} q^{s(2r+1)}$$

$$= \frac{\pi}{2}\left[1 + 2\sum_{n=1}^{\infty} q^{n^2}\right]^2$$

When the real period is much larger than the imaginary period, $q$ is close to 1, and the above expansions do not converge very rapidly.

One may then go back to a ratio of the periods which is more advantageous, with the aid of one of the formulas which permit dividing the ratio of the periods by an odd integer. One may also utilize the above expansions, within their limits of convergence, for imaginary values of $u$ of the form $u = K + iv$ (where $v$ is real) up to $v = K'/2$, then continue them by the formula of displacement of a half period.

## 3. THETA FUNCTIONS

### 3.0 Advantage of the Theta Functions

The elliptic functions and their logarithms are represented by singularities of alternating sense; the necessity for this is clear for those which are real, or have a real constant part on the contour of a rectangle due to the method of analytic continuation by symmetry. This limits the possibilities of calculation by addition of the singularities

in the whole of the plane. The difficulty appeared first for the multiplication of the ratio of the periods of the function $\chi(u)$ by an even integer. It was also clear for the functions $\varphi(u)$ and $\psi(u)$ and, if we had preferred to consider the logarithms of the elliptic functions, represented by sources the strength of which is conserved in the course of the conformal transformations, the difficulty would have continued to exist for the elliptic functions whose singularities are poles, which are just as easy to add as the logarithms, but whose rules of conservation affect the residues, that is, the properties of the logarithms which integrate them.

In order to separate the various singularities which reproduce themselves periodically at the intervals $4K$ and $4iK'$, without associating them automatically with singularities of inverse sense, it is desirable to define a function presenting singularities which are all of the same sense.

After separating the essential singularities, we have only to consider those of the sources and of the poles. These latter are deduced from the sources, whatever their order may be, by differentiation. It suffices therefore to study the sources.

The theta functions are the analytic functions, the logarithms of which are represented by the fields of double infinities of regularly spaced sources at the vertices of rectangles. One theta function is linked to the study of the elliptic functions with real modulus whose ratio of periods is equal to the ratio of the sides of the rectangle defining this theta function. In generalizing according to the rules of reduction of the elliptic functions, it is sufficient that the proportion of these ratios be expressed by a simple fraction, a ratio of two integers.

An elliptic function with complex modulus $k^2$ may likewise be associated with a theta function corresponding to the field of a double infinity of spaced sources with periods the ratio of which ceases to be purely imaginary. One sees that, if the projection of one period on another is a simple fraction, a ratio of two integers, the theta function will be identical with the theta function associated with elliptic functions of real modulus, except as will be discussed later.

The elliptic functions with imaginary modulus have, so far, only infrequently been applied in fluid mechanics, and do not come within the scope of the present study.

### 3.1 Definition of the Theta Functions

The analytic functions are related to the flow of an incompressible fluid, and it is not possible to conceive of an infinite number of sources

without providing for the disposal of their outputs. The theta functions prohibit placing sinks at finite distances, and one must admit the existence of an essential singularity at infinity which characterizes the infinite output of the sources.

It would be possible to reduce the double infinity of the sources to a single one by logarithmic transformation as was done in the preceding chapter for the $\chi(u)$ function, but it is simpler to argue directly in the plane of definition of the theta function for illustrating the multiplicity of determinations of this function.

The field of a simple infinity of sources, regularly spaced on a straight line (fig. 31), is known and corresponds to the analytic function

$$\ln\left[\sin \frac{\pi z}{a}\right]$$

Thus, it is possible to define a theta function by superposition of terms of the form

$$\ln\left[\frac{\sin \pi\left(\frac{z + rib}{a}\right)}{\cosh\left(r\pi \frac{b}{a}\right)}\right]$$

where the denominator $\mathrm{ch}\left(r\pi \frac{b}{a}\right)$ was introduced in such a manner that the logarithm tends toward zero when $r$ increases indefinitely while $z$ and $b/a$ remain constant.

$$\ln \theta = \sum_{r=-\infty}^{r=\infty} \ln\left[\frac{\sin \pi\left(\frac{z + rib}{a}\right)}{\cosh \pi r \frac{b}{a}}\right] + C^{te}$$

$$\theta = C^{te} \times \prod_{r=-\infty}^{r=\infty} \frac{\sin\left(\pi \frac{z + rib}{a}\right)}{\cosh \pi r \frac{b}{a}}$$

However, this shows the possibility of defining another theta function by interchanging the roles of a and b, and of defining an infinity of theta functions – combining the two first ones after various changes of origin and adding functions which do not have any singularity at finite distance like the circular sine and cosine functions. The singularity to be retained at infinity is therefore not actually determined.

It suffices for the intended applications to choose a single function for a given distribution of zeros, and only the first definition given by the above equation will be retained. A simple transformation yields

$$\theta = C^{te} \times \sin\left(\frac{\pi z}{a}\right) \prod_{r=1}^{r=\infty}\left[1 - \frac{\cos^2\left(\pi \frac{z}{a}\right)}{\cosh^2\left(\pi r \frac{b}{a}\right)}\right]$$

The constant is chosen real.

The theta function admits the period $2a$. It is real when $z$ is real and when the real part of $z$ is an odd multiple of $a/2$. It is purely imaginary when the real part of $z$ is an even multiple of $a/2$. Its field is represented by figure 32.

Since the symmetry of the roles of $a$ and $b$ is agreed upon and the scale factor is rather inconvenient, the notations are simplified

$$v = \frac{z}{a} \qquad \tau = \frac{ib}{a} = -\frac{i}{\pi} \ln q \qquad q < 1$$

$$\theta(v,q) = 2C_o \sin \pi v \prod_{r=1}^{\infty} \left[\left(1 - q^{2r} e^{2\pi i v}\right)\left(1 - q^{2r} e^{-2\pi i v}\right)\right]$$

In order to simplify the notation in the calculations which lead frequently to a displacement of the origin by a quarter period, one denotes the value of $\theta$ after such a change of origin by $\theta_1$.

$$\theta_1(v,q) = \theta\left(v + \frac{1}{2},q\right) = 2C_o \cos \pi v \prod_{r=1}^{\infty} \left(1 + q^{2r} e^{2\pi i v}\right)\left(1 + q^{2r} e^{-2\pi i v}\right)$$

Because of the symmetry of the roles of $a$ and $b$, masked by the choice made for the theta function, the calculations lead likewise to utilizing a displacement of the origin by $\tau/2$, but the functions $\theta\left(v + \frac{\tau}{2},q\right)$ and $\theta_1\left(v + \frac{\tau}{2},q\right)$ are not real when $v$ is real.

$$\theta_1\left(v + \frac{\tau}{2},q\right) = C_o\, q^{-\frac{1}{2}} e^{-i\pi v} \prod_{r=0}^{\infty} \left[\left(1 + q^{2r+1} e^{2i\pi v}\right)\left(1 + q^{2r+1} e^{-2\pi i v}\right)\right]$$

The product $\theta_1\left(v + \frac{\tau}{2}, q\right)e^{i\pi v}$ is real when $v$ is real. It will be designated by $\lambda\theta_2(v,q)$ and used in the calculations, in preference to using $\theta_1\left(v + \frac{\tau}{2}, q\right)$.

If the origin is displaced once more by $\tau/2$,

$$\lambda\theta_2\left(v + \frac{\tau}{2}, q\right) = C_o\, q^{-\frac{1}{2}} \left(1 + e^{-2\pi iv}\right) \prod_1^\infty \left[\left(1 + q^{2r} e^{2\pi iv}\right)\left(1 + q^{2r} e^{-2\pi iv}\right)\right]$$

$$= q^{-\frac{1}{2}} e^{-\pi iv}\, \theta_1(v,q)$$

Hence, the two formulas

$$\theta_1\left(v + \frac{\tau}{2}, q\right) = \lambda e^{-\pi iv}\, \theta_2(v,q)$$

$$\theta_2\left(v + \frac{\tau}{2}, q\right) = \frac{q^{-\frac{1}{2}}}{\lambda} e^{-\pi iv}\, \theta_1(v,q)$$

In order to complete the symmetry of these two formulas, one chooses $\lambda = q^{-1/4}$.

Elimination of $\theta_2$ after change of $v$ and $v + \frac{\tau}{2}$ in the first formula furnishes

$$\theta_1(v + \tau, q) = q^{-1} e^{-2i\pi v}\, \theta_1(v,q)$$

Finally, a $\theta_3$ function can be defined by

$$\theta_3(v,q) = \theta_2\left(v + \frac{1}{2}, q\right)$$

The functions $\ln \theta_2$ and $\ln \theta_3$ differ only by the origin of the complex variable and are represented by the same field (fig. 33).

Finally, the theta function and the auxiliary functions $\theta_1$, $\theta_2$, and $\theta_3$ may be defined by the infinite products

$$\left.\begin{aligned}
\theta(v,q) &= 2C_o \sin \pi v \prod_{r=1}^{\infty} \left[\left(1 - q^{2r} e^{2\pi i v}\right)\left(1 - q^{2r} e^{-2\pi i v}\right)\right] \\
\theta_1(v,q) &= 2C_o \cos \pi v \prod_{r=1}^{\infty} \left[\left(1 + q^{2r} e^{2\pi i v}\right)\left(1 + q^{2r} e^{-2\pi i v}\right)\right] \\
\theta_2(v,q) &= C_o\, q^{-\frac{1}{4}} \prod_{r=0}^{\infty} \left[\left(1 + q^{2r+1} e^{2\pi i v}\right)\left(1 + q^{2r+1} e^{-2\pi i v}\right)\right] \\
\theta_3(v,q) &= C_o\, q^{-\frac{1}{4}} \prod_{r=0}^{\infty} \left[\left(1 - q^{2r+1} e^{2\pi i v}\right)\left(1 - q^{2r+1} e^{-2\pi i v}\right)\right]
\end{aligned}\right\}$$

with the formulas of change of origin by a quarter period, written without mention of the parameter $q$

$$\left.\begin{aligned}
\theta\left(v + \tfrac{1}{2}\right) &= \theta_1(v) & \qquad \theta_1\left(v + \tfrac{1}{2}\right) &= -\theta(v) \\
\theta_2\left(v + \tfrac{1}{2}\right) &= \theta_3(v) & \qquad \theta_3\left(v + \tfrac{1}{2}\right) &= \theta_2(v)
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\theta\left(v + \tfrac{\tau}{2}\right) &= iq^{-\frac{1}{4}} e^{-i\pi v}\, \theta_3(v) & \qquad \theta_3\left(v + \tfrac{\tau}{2}\right) &= iq^{-\frac{1}{4}} e^{-i\pi v}\, \theta(v) \\
\theta_1\left(v + \tfrac{\tau}{2}\right) &= q^{-\frac{1}{4}} e^{-i\pi v}\, \theta_2(v) & \qquad \theta_2\left(v + \tfrac{\tau}{2}\right) &= q^{-\frac{1}{4}} e^{-i\pi v}\, \theta_1(v)
\end{aligned}\right\}$$

and for a displacement by a half period

$$\left.\begin{aligned}
\theta(v + 1) &= -\theta(v) & \qquad \theta_1(v + 1) &= -\theta_1(v) \\
\theta_2(v + 1) &= \theta_2(v) & \qquad \theta_3(v + 1) &= \theta_3(v)
\end{aligned}\right\}$$

$$\left.\begin{aligned}\theta(v+\tau) &= -q^{-1}e^{-2i\pi v}\,\theta(v) & \theta_3(v+\tau) &= -q^{-1}e^{-2i\pi v}\,\theta_3(v)\\ \theta_1(v+\tau) &= q^{-1}e^{-2i\pi v}\,\theta_1(v) & \theta_2(v+\tau) &= q^{-1}e^{-2i\pi v}\,\theta_2(v)\end{aligned}\right\}$$

The last formula may be interpreted. The function $\theta_2(v)e^{i\pi\frac{v^2}{\tau}}$ admits the period $\tau$. It is real when the variable $v$ is real or purely imaginary. It corresponds to the interchange of the roles of $a$ and $b$ in the initial notation and its logarithm is represented by figure 34.

This remark may be made more specific:

Each one of the terms of $\theta_2(v,q)$ admits itself an infinity of zeros and can be represented by an infinite product

$$1 + q^{2r+1}e^{2\pi iv} = \left(1 + q^{2r+1}\right)\prod_{s=-\infty}^{\infty}\left[1 - \frac{2v}{(2s+1) - i(2r+1)\frac{K'}{K}}\right]$$

Hence, the purely formal expression

$$\theta_2\left(v,e^{-\frac{\pi K'}{K}}\right) = \theta_2\left(0,e^{-\frac{\pi K'}{K}}\right)\prod_{s=-\infty}^{\infty}\prod_{r=-\infty}^{\infty}\left[1 - \frac{2v}{(2s+1) - i(2r+1)\frac{K'}{K}}\right]$$

This product is not convergent, but an artifice that consists in arguing in terms of a derivative of sufficiently high order that the coefficients of the factors tend rapidly toward zero when $r$ or $s$ increase. It is more convenient to derive a sum by calculating

$$\frac{d^3}{dv^3}\left[\ln\left[\theta_2\left(v,e^{-\frac{\pi K'}{K}}\right)\right]\right] = -2\sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\left[\frac{\dfrac{2}{(2s+1)+i(2r+1)K'/K}}{1 - \dfrac{2v}{(2s+1)-i(2r+1)K'/K}}\right]$$

This sum is convergent and shows that the function $\theta_2\left(\frac{vK}{K'},e^{-\frac{\pi K}{K'}}\right)$ which corresponds to the interchange of $a$ and $b$ in the definition of $\theta_2$ is related to the original function by

$$\frac{d^3}{dv^3}\left[\ln\left[\theta_2\left(\frac{vK}{K'}, e^{-\frac{\pi K}{K'}}\right)\right]\right] = \left(\frac{iK'}{K}\right)^3 \frac{d^3}{dv^3}\left[\ln\left[\theta_2\left(iv, e^{-\frac{\pi K'}{K}}\right)\right]\right]$$

The integration furnishes

$$\theta_2\left(\frac{vK}{K'}, e^{-\frac{\pi K}{K'}}\right) = \theta_2\left(iv, e^{-\frac{\pi K'}{K}}\right) e^{Av^2+Bv+C}$$

The periodicity observed above leads to replacing $v$ by $v + \frac{K'}{K}$ in this relation. The formulas for change of origin then furnish

$$A = -\frac{\pi K}{K'} \qquad B = 0$$

The constant $C$ may be specified for $v = 0$

$$\frac{\theta_2\left(iv, e^{-\frac{\pi K'}{K}}\right)}{\theta_2\left(0, e^{-\frac{\pi K'}{K}}\right)} = e^{\frac{\pi K}{K'}v^2} \frac{\theta_2\left(\frac{vK}{K'}, e^{-\frac{\pi K}{K'}}\right)}{\theta_2\left(0, e^{-\frac{\pi K}{K'}}\right)}$$

Changes of origin permit deducing from the above formula the group of formulas for change of axes

$$\frac{\theta_1\left(iv, e^{-\frac{\pi K'}{K}}\right)}{\theta_2\left(0, e^{-\frac{\pi K'}{K}}\right)} = e^{\frac{\pi K}{K'}v^2} \frac{\theta_3\left(\frac{vK}{K'}, e^{-\frac{\pi K}{K'}}\right)}{\theta_2\left(0, e^{-\frac{\pi K}{K'}}\right)}$$

$$\frac{\theta_3\left(iv, e^{-\frac{\pi K'}{K}}\right)}{\theta_2\left(0, e^{-\frac{\pi K'}{K}}\right)} = e^{\frac{\pi K}{K'}v^2} \frac{\theta_1\left(\frac{vK}{K'}, e^{-\frac{\pi K}{K'}}\right)}{\theta_2\left(0, e^{-\frac{\pi K}{K'}}\right)}$$

$$\frac{\theta\left(iv, e^{-\frac{\pi K'}{K}}\right)}{\theta_2\left(0, e^{-\frac{\pi K'}{K}}\right)} = ie^{\frac{\pi K}{K'}v^2} \frac{\theta\left(\frac{vK}{K'}, e^{-\frac{\pi K}{K'}}\right)}{\theta_2\left(0, e^{-\frac{\pi K}{K'}}\right)}$$

## 3.2 Expression of the Elliptic Functions and of Their Logarithms With the Aid of the Theta Functions

An elliptic function is a rational fraction of sn u, cn u, dn u. Let m be the degree of its numerator and n the degree of its denominator with respect to the whole of these functions. Except for the case of reduction, there exist, in a rectangle of 4K width and 4iK' height, 4 m zeros and 4 n poles which can be determined as roots of polynomials. The numerator, for instance, may be written in the form

$$G_m(\text{sn } u, \text{cn } u) + \text{dn } u \, H_{m-1}(\text{sn } u, \text{cn } u)$$

its roots are among those of

$$G_m{}^2(\text{sn } u, \text{cn } u) - \left(1 - k^2 \text{sn}^2 u\right) H_{m-1}{}^2(\text{sn } u, \text{cn } u)$$

This function may be written in the form

$$g_{2m}(\text{sn } u) + \text{cn } u \, h_{2m-1}(\text{sn } u)$$

its roots are among those of

$$g_{2m}{}^2(\text{sn } u) - \left(1 - \text{sn}^2 u\right) h_{2m-1}{}^2(\text{sn } u)$$

This polynomial of degree 4m in sn u has four m roots and every root defines four zeros in a rectangle of the periods. Among these 16 m zeros, only four m are zeros of the numerator of the elliptic function since the method that was followed furnishes the zeros of

$$G_m(\text{sn } u, \pm \text{ cn } u) \pm \text{dn } u \, H_{m-1}(\text{sn } u, \pm \text{ cn } u) = 0$$

The determination of the roots of a polynomial of high degree can be difficult unless it is only a matter of numerical calculations. Once that determination is made, it is easy to separate the convenient roots.

If the degree m of the numerator differs from the degree n of the denominator, the elliptic function presents four poles of the order m - n or four zeros of the order n - m, with the common poles sn u, cn u, dn u. Finally, the function presents as many poles as zeros, if each one is counted with its order.

The logarithm of the elliptic function is represented by sources, the intensity of which is the product of $2\pi$ and the order of the zero or the order of the pole, counted negatively. It appears, intuitively, that this logarithm is defined by these singularities like a sum of logarithms of theta functions. To demonstrate this, it is just as easy to reason directly regarding the elliptic function $F(u)$ which can then be written

$$F(u) = \frac{\theta\left(\frac{u - a_1}{4K}\right) \times \ldots \times \theta\left(\frac{u - a_p}{4K}\right)}{\theta\left(\frac{u - b_1}{4K}\right) \times \ldots \times \theta\left(\frac{u - b_p}{4K}\right)} f(u)$$

where $a_1, \ldots, a_p$ are the zeros, $b_1, \ldots, b_p$ the poles after eventual reduction of the poles common to the numerator and denominator. If multiple poles or zeros exist, the corresponding theta function appears with a degree equal to the order. Writing that $F(u)$ admits the periods $4K$ and $4iK'$, one will obtain, according to the formulas for displacement of the origin of the theta functions

$$f(u + 4K) = f(u) \qquad f(u + 4iK') = f(u)e^{i\frac{\pi}{2K}(r+is)}$$

with

$$r + is = b_1 + \ldots + b_p - a_1 \ldots -a_p$$

The function $f(u)$ has neither pole nor zero at a finite distance. Its logarithm $g(u)$ has no singularity at a finite distance and follows the simpler law

$$g(u + 4K) = g(u) + 2i\pi\lambda$$

$$g(u + 4iK') = g(u) + \frac{i\pi}{2K} (r + is) + 2i\pi\mu$$

where $\lambda$ and $\mu$ are integers.

Neither has the derivative $dg/du$ a singularity at a finite distance, and since the derivation of the above formulas shows that it is doubly periodic, its continuation to infinity does not introduce any singularity at infinity. It can therefore only be a constant. Consequently

$$g(u) = Au + B$$

where A and B are constants.

Using this expression in the law for displacement of the periods for $g(u)$, one finds

$$r + is = -4\mu K + 4i\lambda K' \qquad A = i\pi\lambda/2K$$

The sum of the complex variables of the poles minus the sum of the complex variables of the zeros is a multiple of the periods which certainly proves that it is not possible to represent by an elliptic function a function defined by arbitrary singularities as was indicated in section 2.1.

Even though the zeros and the poles are defined except for integer multiples of the periods, it is still possible to choose them in such a manner that $r + is$, is zero. In this case A is zero, $g(u)$ and $f(u)$ are constants:

$$F(u) = \frac{\theta\left(\frac{u - a_1}{4K}\right) \times \ldots \times \theta\left(\frac{u - a_p}{4K}\right)}{\theta\left(\frac{u - b_1}{4K}\right) \times \ldots \times \theta\left(\frac{u - b_p}{4K}\right)} \times C^{te}$$

The constant is to be determined for an arbitrary value of the variable u. The logarithm of the elliptic function $F(u)$ is simultaneously determined.

In the simple cases where the zeros and poles are obvious, it is frequently more convenient not to try to make $r + is$ zero but to replace the constant by

$$C^{te} \times e^{i\pi\lambda \frac{u}{2K}}$$

Finally, if four zeros or four poles differ only by half periods, one will be able to replace the product of the four theta functions by $\theta\left(\frac{u - a}{2K}\right)$.

Let us apply to the function $\operatorname{sn} u$ which presents the zeros of $\theta\left(\frac{u}{2K}\right)$ and whose poles are the zeros of $\theta\left(\frac{u - iK'}{2K}\right)$

$$\frac{\operatorname{sn} u}{\operatorname{sn} u_o} = \frac{\theta\left(\frac{u}{2K}\right)}{\theta\left(\frac{u - iK'}{2K}\right)} \times \frac{\theta\left(\frac{u_o - iK'}{2K}\right)}{\theta\left(\frac{u_o}{2K}\right)} \times e^{i\pi\lambda\frac{u-u_o}{2K}}$$

or else, choosing $u_o = K$, expressing $\theta$ as a function of $\theta_3$, and noting that $\operatorname{sn} u$, $\theta$, $\theta_3$ are real with $u$

$$\operatorname{sn} u = \frac{\theta\left(\frac{u}{2K}\right)}{\theta_3\left(\frac{u}{2K}\right)} \times \frac{\theta_3\left(\frac{1}{2}\right)}{\theta\left(\frac{1}{2}\right)} = \frac{\theta\left(\frac{u}{2K}\right)}{\theta_3\left(\frac{u}{2K}\right)} \times \frac{\theta_2(o)}{\theta_1(o)}$$

Likewise

$$\operatorname{cn} u = \frac{\theta_1\left(\frac{u}{2K}\right)}{\theta_3\left(\frac{u}{2K}\right)} \times \frac{\theta_3(o)}{\theta_1(o)} \qquad \operatorname{dn} u = \frac{\theta_2\left(\frac{u}{2K}\right)}{\theta_3\left(\frac{u}{2K}\right)} \times \frac{\theta_3(o)}{\theta_2(o)}$$

From the last relation, written for $u = K$, there results

$$k' = \cos \Theta = \left[\frac{\theta_3(o)}{\theta_2(o)}\right]^2 = \left[\prod_{r=0}^{\infty} \frac{1 - q^{2r+1}}{1 + q^{2r+1}}\right]^4$$

From the relation pertaining to $\operatorname{sn} u$, written for $u = K + iK'$, results

$$k = \sin \Theta = \left[\frac{\theta_1(o)}{\theta_2(o)}\right]^2 = 4q^{\frac{1}{2}}\left[\frac{\prod_{r=1}^{\infty} (1 + q^{2r})}{\prod_{r=0}^{\infty} \left(1 + q^{2r+1}\right)}\right]^4$$

These formulas may be utilized for calculation of $\Theta$, jointly with the formulas of 2.6. Other, more convenient formulas will be set up further on. As in 2.6, when $K' < K$, one may interchange the roles of the two periods, setting

$$q' = e^{-\frac{\pi K}{K'}} \qquad \ln q' \ln q = -\pi^2$$

which permits still using the above formulas for a value of $q$ at most equal to $e^{-\pi}$.

The functions $\varphi$, $\chi$, $\psi$ may be defined by

$$\tan \varphi = \frac{\theta_2(o)}{\theta_3(o)} \frac{\theta\left(\frac{u}{2K}\right)}{\theta_1\left(\frac{u}{2K}\right)}$$

$$\tan \chi = \frac{\theta_1(o)}{\theta_3(o)} \frac{\theta\left(\frac{u}{2K}\right)}{\theta_2\left(\frac{u}{2K}\right)}$$

$$\tanh \psi = \frac{\theta_1{}^2(o)}{\theta_2(o)\theta_3(o)} \frac{\theta_1\left(\frac{u}{2K}\right)}{\theta_2\left(\frac{u}{2K}\right)}$$

## 3.3 Expansions of the Theta Functions in Trigonometric Series

The logarithms of the $\theta_2$ and $\theta_3$ functions may be expanded in trigonometric series like the $\varphi$, $\chi$, $\psi$ functions, but it is more advantageous to directly expand the theta functions in trigonometric series.

Let the function be, for instance, $\theta_2$ which, due to its being even and to its periodicity, may be written, for a convenient value of the constant $C_o$ left undetermined in section 3.

$$\theta_2(v,q) = 1 + 2\sum_{s=1}^{\infty} \alpha_s \cos 2s\pi v = +\sum_{s=-\infty}^{\infty} \alpha_s\, e^{2si\pi v}$$

Inserting this expression into the relation

$$\theta_2\left(v + \frac{\tau}{2}, q\right) = q^{-1}\, e^{-2i\pi v}\, \theta_2(v,q)$$

and setting the terms equal, one will find

$$\alpha_s = q^{2s-1}\, \alpha_{s-1} = q^{(2s-1)+(2s-3)+\ldots+1} = q^{s^2}$$

and consequently

$$\theta_2(v,q) = 1 + 2\sum_{s=1}^{\infty} q^{s^2} \cos 2s\pi v$$

$$\theta_3(v,q) = \theta_2\left(v + \frac{1}{2},q\right) = 1 + 2\sum_{s=1}^{\infty} (-1)^s q^{s^2} \cos 2s\pi v$$

Likewise

$$\theta_1(v,q) = q^{\frac{1}{4}} e^{i\pi v}\, \theta_2\left(v + \frac{1}{2},q\right)$$

$$= \sum_{s=-\infty}^{\infty} q^{s^2+s+\frac{1}{4}} e^{(2s+1)i\pi v}$$

$$= 2\sum_{s=0}^{\infty} q^{\left(s+\frac{1}{2}\right)^2} \cos(2s + 1)\pi v$$

$$\theta(v,q) = -\theta_1\left(v + \frac{1}{2},q\right)$$

$$= 2\sum_{s=0}^{\infty} (-1)^s q^{\left(s+\frac{1}{2}\right)^2} \sin(2s + 1)\pi v$$

The value of $C_o$ resulting from these expressions will be retained hereafter. Setting, for instance, the two values of $\theta_1(o)$ equal:

$$C_o = \frac{\sum_{s=0}^{\infty} q^{\left(s+\frac{1}{2}\right)^2}}{\left[\prod_{r=1}^{\infty} (1 + q^{2r})\right]^2}$$

The trigonometric series of the theta functions are very convergent and convenient for the calculation of these functions and of the elliptic functions. Unfortunately, the theta functions do not satisfy any simple rule of addition permitting calculation of their values for complex values with the aid of their values for real values.

The values of $k$ and $k'$ corresponding to a value of $q$ may be calculated by the formulas

$$k = \left[\frac{\theta_1(o)}{\theta_2(o)}\right]^2 = \left[\frac{2\sum_{o}^{\infty} q^{\left(s+\frac{1}{2}\right)^2}}{1 + 2\sum_{1}^{\infty} q^{s^2}}\right]^2$$

$$k' = \left[\frac{\theta_3(o)}{\theta_2(o)}\right]^2 = \left[\frac{1 + 2\sum_{1}^{\infty}(-1)^s q^{s^2}}{1 + 2\sum q^{s^2}}\right]^2$$

which result immediately from the expressions of the functions sn u, cn u, dn u, with the aid of the theta functions which have been established in the preceding paragraph.

These formulas are retained for the calculation of $k$ and $k'$, in preference to the formulas of 2.6 and 3.2.

One will note, moreover, that the formula established in 2.6 for the calculation of the quarter period $K$ is written

$$\frac{2K}{\pi} = \left[\theta_2(o)\right]^2$$

## 3.4 Elliptic Functions With Given Zeros and Poles

If the zeros $a_1 \ldots a_p$ and the poles $b_1 \ldots b_p$ of an elliptic function are given, obviously, for convenience, satisfying the relation:

$$a_1 + \ldots a_p = b_1 + \ldots + b_p$$

the function is defined by

$$\frac{F(u)}{F(u_o)} = \frac{\theta\left(\frac{u - a_1}{4K}\right)\ldots\theta\left(\frac{u - a_p}{4K}\right)}{\theta\left(\frac{u - b_1}{4K}\right)\ldots\theta\left(\frac{u - b_p}{4K}\right)}\,\frac{\theta\left(\frac{u_o - b_1}{4K}\right)\ldots\theta\left(\frac{u_o - a_p}{4K}\right)}{\theta\left(\frac{u_o - b_1}{4K}\right)\ldots\theta\left(\frac{u_o - a_p}{4K}\right)}$$

where $F(u_o)$ is an arbitrary constant chosen as the value of the function in a complex variable $u_o$ different from those of the zeros and the poles. The problem is to show that it is possible to express this function with the aid of the three functions sn u, cn u, dn u.

Let us begin with the simple function admitting two zeros and two poles. The relation

$$a_1 + a_2 = b_1 + b_2$$

expresses that the logarithm of the function is represented by two sources and two sinks at the vertices of a parallelogram. One may also regard it as corresponding to the superposition of the field of a sink at the center of the parallelogram which is capable of absorbing the output of the two sources and of the field of one source feeding the two sinks. Besides, if the problem is solved when the center of the parallelogram is at the origin, the formulas of addition will permit a change in origin. Thus, it suffices to study

$$\frac{\theta\left(\frac{u - u_1}{4K}\right)\theta\left(\frac{u + u_1}{4K}\right)}{\theta^2\left(\frac{u}{4K}\right)}$$

This function has the same singularities as $\frac{1}{\mathrm{sn}^2 \frac{u_1}{2}} - \frac{1}{\mathrm{sn}^2 \frac{u}{2}}$ and, choosing as a reference $u_o = 2iK'$

$$\mathrm{sn}^2 \frac{u_1}{2}\left[\frac{1}{\mathrm{sn}^2 \frac{u_1}{2}} - \frac{1}{\mathrm{sn}^2 \frac{u}{2}}\right] = \frac{\theta\left(\frac{u - u_1}{4K}\right)\theta\left(\frac{u + u_1}{4K}\right)}{\theta^2\left(\frac{u}{4K}\right)} \frac{\theta^2\left(\frac{\tau}{2}\right)}{\theta\left(\frac{\tau}{2} - \frac{u_1}{4K}\right)\theta\left(\frac{\tau}{2} + \frac{u_1}{4K}\right)}$$

which becomes after reduction

$$\frac{1}{\mathrm{sn}^2 \frac{u_1}{2}} - \frac{1}{\mathrm{sn}^2 \frac{u}{2}} = \frac{\theta_1{}^2(o)\theta_3{}^2(o)}{\theta_2{}^2(o)} \frac{\theta\left(\frac{u - u_1}{4K}\right)\theta\left(\frac{u + u_1}{4K}\right)}{\theta^2\left(\frac{u}{4K}\right)\theta^2\left(\frac{u_1}{4K}\right)}$$

on the other hand, the formulas of addition establish

$$\mathrm{sn}^2 \frac{u}{2} = \frac{1 - \mathrm{cn}\, u}{1 + \mathrm{dn}\, u}$$

The problem is thus solved for the elementary elliptic function retained.

Let us go back to the general problem and suppose that the number of the zeros and of the poles $p$ is a power of two. If this is not the case, it will still be possible to multiply top and bottom parts by the same theta functions.

Let us then group the terms two by two:

$$\theta\left(\frac{u - a_1}{4K}\right)\theta\left(\frac{u - a_2}{4K}\right) = \theta^2\left(u - \frac{a_1 + a_2}{8K}\right)f(u)$$

where $f(u)$ is expressed as a function of $\operatorname{cn} u$, $\operatorname{sn} u$, $\operatorname{dn} u$.

Likewise

$$\theta\left(u - \frac{a_1 + a_2}{8K}\right)\theta\left(u - \frac{a_1 + a_2}{8K}\right) = \theta^2\left(u - \frac{a_1 + a_2 + a_3 + a_4}{16K}\right)g(u)$$

Since the sum of the complex variables of the roots is equal to the sum of the complex variables of the poles, the method eliminates the theta functions, and a rational function of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ remains.

## 3.5 Use of Theta Functions

Theta functions appear useful for the calculation of the elliptic functions and for the investigation of elliptic functions with given singularities.

They permit establishing more simply the properties of the elliptic functions demonstrated in the preceding chapter. All demonstrations were, in fact, based on the possibility of addition of singularities, and it is much more convenient to isolate these singularities rather than search for combinations permitting compensation of the parasitic singularities. The formulas for multiplication or division of the ratio of the periods by an even or odd integer are written very simply

$$\frac{\theta(v,q)}{\theta_1(o,q)} = \frac{\theta\left(\frac{v}{r}, q^{\frac{1}{r}}\right)\theta\left(\frac{v+1}{r}, q^{\frac{1}{r}}\right) \dots \theta\left(\frac{v+r-1}{r}, q^{\frac{1}{r}}\right)}{\theta\left(\frac{1}{2r}, q^{\frac{1}{r}}\right)\theta\left(\frac{3}{2r}, q^{\frac{1}{r}}\right) \dots \theta\left(1 - \frac{1}{2r}, q^{\frac{1}{r}}\right)}$$

and

$$\frac{\theta(v,q)}{\theta_1(o,q)} = \frac{\theta\left(v,q^r\right)\theta\left(v + \tau, q^r\right) \dots \theta\left[v + (r-1)\tau, q^r\right]}{\theta_1\left(o,q^r\right)\theta_1(\tau,q^r) \dots \theta_1\left[(r-1)\tau, r^2\right]} e^{\pi i(r-1)\left(v - \frac{1}{2}\right)}$$

Finally, the theta functions permit calculation of the elliptic integrals as will be shown in the following chapter.

We note that it would have been possible to study the theta functions before the elliptic functions, but the method would have been particularly for these uniform functions with double periodicity, and would have concealed the generalization to the Schwarz functions.

## 4. ZETA FUNCTIONS AND CALCULATION OF THE ELLIPTIC INTEGRALS

### 4.0 Definition of the Zeta Functions

The theta functions were introduced, in the preceding chapter, by their logarithms, represented by fields of sources analogous to the fields of sources and sinks of the logarithms of the elliptic functions. It appeared there convenient to use the theta functions for the calculations, in preference to their logarithms; however, it is better to study the latter whose derivatives are simpler, in the present chapter.

A zeta function is the derivative of the logarithm of a theta function

$$Z(v,q) = \frac{d}{dv} \ln\left[\theta(v,q)\right]$$

and likewise $Z_1$, $Z_2$, $Z_3$[1] (capital $\zeta$) for $\theta_1$, $\theta_2$, $\theta_3$.

Derivation of the relationships

$$\ln\left[\theta(v+1,q)\right] = \ln\left[-\theta(v,q)\right]$$

$$\ln\left[\theta(v+\tau,q)\right] = \ln\left[\theta(v,q)\right] - 2i\pi\left(v+\frac{\tau}{2}\right)$$

establishes for the zeta function

$$Z(v+1,q) = Z(v,q)$$

$$Z(v+\tau,q) = Z(v,q) - 2i\pi$$

The zeta function admits the real period 1 and, except for a multiple of $2i\pi$, the imaginary period $\tau$.

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[1] The zeta function thus defined is linked to the function $Zn(u,q)$ of Jacobi by $Z_3(v,q) = 2KZn(2Kv,q)$. It seemed here more convenient to choose the logarithmic derivative of the theta function with respect to its normal variable $v$ rather than with respect to the normal variable of the elliptic functions $u$ which is sometimes $4Kv$.

Likewise, the differentiation of the relation

$$\ln\left[\theta\left(v + \frac{\tau}{2}, q\right)\right] = \ln\left[i\theta_3(v,q)\right] - \pi i\left(v + \frac{\tau}{4}\right)$$

given

$$Z\left(v + \frac{\tau}{2}, q\right) = Z_3(v,q) - \pi i$$

shows that, when $v$ is real (with $Z_3(v,q)$ being real), the function $Z\left(v + \frac{\tau}{2}, q\right)$ has a constant imaginary part.

On the other hand, the function $Z(v,q)$ admits a pole at each zero of $\theta(v,q)$. It is purely imaginary when the real part of $v$ is -1/2, 0, 1/2 since $\ln \theta$ has, on these straight lines, a constant imaginary part when $dv$ is imaginary. The representative field is that of figure 35.

## 4.1 Properties of the Zeta Functions

In the neighborhood of one of the zeros of $\theta$, for instance $v_1$, the function $\ln \theta$ behaves like $\ln\left(v - v_1\right) + C^{te}$, and $Z$ behaves like $1/\left(v - v_1\right)$. It admits therefore a double infinity of simple poles of the same sense, in contrast to the elliptic functions which have singularities of alternating sense. Just as it has been possible to define the logarithm of an elliptic function by superposition of logarithms of theta functions, it will be possible to define the elliptic function directly by superposition of zeta functions.

Before examining this essential utilization of the zeta functions, it is of interest to study some of their elementary properties.

If the zeta function represented in figure 35 is derived graphically, the field obtained will present a double pole at the origin, the derivative will be real for the real values of $v$ and for those values of $v$ which have $\tau/2$ as their imaginary part since zeta and $dv$ are real. It will also be real on the straight lines corresponding to the real parts of $v$: -1/2, 0, 1/2, since both $dZ$ and $dv$ are imaginary on these straight lines. The field is that of figure 36. It is defined at the interior of rectangles on the contour of which it represents a real function and corresponds to an elliptic function.

This function is, except for one constant and one factor, $1/\text{sn}^2 2Kv$ or, in other words, the function $p(2Kv)$ of Weierstrass. We shall nevertheless prefer to evaluate the derivative of the function $Z_3(v,q)$ which,

as we have seen, differs from $Z(v,q)$ only by the origin and one constant (fig. 37).

The field is that of a double pole at the center of a rectangle representing, always, except for one coefficient and one factor, the function $dn^2 2Kv$.

$$\frac{d}{dv}\left[Z_3(v,q)\right] = A\ dn^2(2Kv) + B$$

which by integration becomes

$$Z_3(v,q) = C + Bv + A\int_0^v \left[dn^2(2Kv)\right]dv$$

In the neighborhood of $v = \tau/2$, the $Z_3$ function behaves like $\frac{1}{v - \frac{\tau}{2}}$ and its derivative like $-\frac{1}{\left(v - \frac{\tau}{2}\right)^2}$. On the other hand, $dn(2Kv)$ behaves like $\frac{i}{2K\left(v - \frac{\tau}{2}\right)}$ and consequently $A = 4K^2$.

The $Z_3$ function admits the real period 1, and, writing for simplification of the notation

$$E = \int_0^{1/2} \left[dn^2(2Kv)\right] 2K\ dv$$

we shall find, writing that $Z_3(v + 1,q) = Z_3(v,q)$, that the constant $B$ is $-4KE$. The origin of the symbol $E$ will appear in 4.4 when the elliptic integral of the second kind is studied.

Continuing the integration

$$\ln\frac{\theta_3(v,q)}{\theta_2(o,q)} = Cv + 4K^2\int_0^v dv_1\int_0^{v_1}\left[dn^2\left(2Kv_2\right) + E/K\right] \times dv_2$$

and writing that $\theta_3$ admits equally the real period 1, we shall find that $C = 0$. Consequently

$$Z_3(v,q) = 4K^2 \int_0^v \left[dn^2(2Kv_1) - \frac{E}{K}\right] \times dv_1$$

$$\ln \frac{\theta_3(v,q)}{\theta_3(o,q)} = 4K^2 \int_0^v dv_1 \int_0^{v_1} \left[dn^2(2Kv_2) - \frac{E}{K}\right] \times dv_2$$

and for $v = 1/2$

$$\ln \frac{\theta_2(o,q)}{\theta_3(o,q)} = 4K^2 \int_0^{1/2} dv_1 \int_0^{v_1} \left[dn^2(2Kv_2) - \frac{E}{K}\right] \times dv_2$$

Besides this property of its derivative, the zeta function has the advantage of satisfying a law of addition. The logarithmic differentiation

$$\frac{\theta(v + v_1)\theta(v - v_1)}{\theta^2(v)\theta^2(v_1)} = \frac{\theta_2{}^2(o)}{\theta_1{}^2(o)\theta_3{}^2(o)}\left[\frac{1}{sn^2(2Kv_1)} - \frac{1}{sn^2(2Kv)}\right]$$

furnishes in fact

$$Z(v + v_1) + Z(v - v_1) - 2Z(v) = \frac{4K\ cn(2Kv)dn(2Kv)sn^2(2Kv_1)}{sn(2Kv)\left[sn^2(2Kv) - sn^2(2Kv_1)\right]}$$

Interchanging the roles of $v$ and $v_1$ and noting that $Z$ is odd, one has

$$Z(v + v_1) - Z(v - v_1) - 2Z(v_1) = \frac{4K\ cn(2Kv_1)dn(2Kv_1)sn^2(2Kv)}{sn(2Kv_1)\left[sn^2(2Kv_1) - sn^2(2Kv)\right]}$$

and by elimination of $Z(v - v_1)$

$$Z(v + v_1) = Z(v) + Z(v_1) + 2K\frac{cn(2Kv)dn(2Kv)sn^3(2Kv_1) - cn(2Kv_1)dn(2Kv_1)sn^3(2Kv)}{sn(2Kv)sn(2Kv_1)\left[sn^2(2Kv) - sn^2(2Kv_1)\right]}$$

This is a law of addition for $Z(v)$ analogous to that of the elliptic functions.

Applying it to $v_1 = \frac{\tau}{2}$, gives

$$Z_3(v) = Z(v) - 2K \frac{cn(2Kv)dn(2Kv)}{sn(2Kv)}$$

and this formula relates simply $Z_3$ to $Z$ whereas $Z_1$ and $Z_2$ are related still more simply to $Z$ and $Z_3$ by a simple change of origin.

Making a change of origin of $\tau/2$ in the formula of addition of $Z$

$$Z_3\left(v + v_1\right) = Z_3(v) + Z_3\left(v_1\right) - 2Kk^2 sn(2Kv)sn\left(2Kv_1\right)sn\left[2K\left(v + v_1\right)\right]$$

This formula of addition is simple. In order to be able to apply it to the calculation of $Z_3$ when $v$ is real and $v_1$ purely imaginary, one must have available a means for calculation of $Z_3(iv)$.

The relation between the values of the zeta function for the imaginary values and for the real values of the variable result from the corresponding relationship between the theta functions established in 3.1

$$\ln\left[\theta_2\left(iv,e^{i\pi\tau}\right)\right] = +i\frac{\pi v^2}{\tau} + \ln\left[\theta_2\left(+\frac{iv}{\tau},e^{-\frac{i\pi}{\tau}}\right)\right] + C^{te}$$

which furnishes by differentiation

$$Z_2\left(iv,e^{i\pi\tau}\right) = +\frac{2\pi v}{\tau} + \frac{1}{\tau} Z_2\left(+\frac{iv}{\tau},e^{-\frac{i\pi}{\tau}}\right)$$

and by a displacement of origin replacing $v$ by $v - \frac{i}{2}$

$$Z_3\left(iv,e^{i\pi\tau}\right) = \frac{2\pi v}{\tau} + \frac{1}{\tau} Z_1\left(\frac{iv}{\tau},e^{-\frac{i\pi}{\tau}}\right)$$

For practical purposes, it is more convenient to use the relation in the form

$$Z_3\left[iv + \frac{1}{2},e^{i\pi\tau}\right] = \frac{2\pi v}{\tau} + \frac{1}{\tau} Z_3\left[\frac{iv}{\tau} + \frac{1}{2},e^{-\frac{i\pi}{\tau}}\right]$$

for calculating

$$Z_3\left[\alpha + i\beta, e^{i\pi\tau}\right] = Z_3\left[\alpha - \frac{1}{2} + i\beta + \frac{1}{2}, e^{i\pi\tau}\right]$$

$$= Z_3\left[\alpha - \frac{1}{2}, e^{i\pi\tau}\right] + \frac{2\pi\beta}{\tau} + \frac{1}{\tau} Z_3\left[\frac{i\beta}{\tau} + \frac{1}{2}, e^{-\frac{i\pi}{\tau}}\right] -$$

$$2Kk^2 \operatorname{sn}\left[2K\left(\alpha - \frac{1}{2}\right)\right] \operatorname{sn}\left[2K\left(i\beta + \frac{1}{2}\right)\right] \operatorname{sn}\left[2K(\alpha + i\beta)\right]$$

where the $Z_3$ functions appearing in the second term correspond to real values of the variables when $\alpha$ and $\beta$ are real. The calculation of the $Z_3$ function for the complex values of the variable is therefore reduced to the calculation of the $Z_3$ function for the real values of the variable. The other zeta functions are simply reduced to $Z_3$.

From the above formulas there results an important relation between $E$, $E'$, $K$, $K'$

$$\frac{d}{dv}\left[Z_3\left(iv + \frac{1}{2}, e^{i\pi\tau}\right)\right] = \frac{2\pi}{\tau} + \frac{1}{\tau}\frac{d}{dv}\left[Z_3\left(\frac{iv}{\tau} + \frac{1}{2}, e^{-\frac{i\pi}{\tau}}\right)\right]$$

and according to the formula for differentiation of $Z_3$

$$4iK^2\left[\operatorname{dn}^2(2iKv + K,k) - \frac{E}{K}\right] = \frac{2\pi}{\tau} + \frac{1}{\tau^2} \times 4K'^2\left[\operatorname{dn}^2(2K'v + K',k') - \frac{E'}{K'}\right]$$

the laws of addition show that this formula is independent of $v$ and is written

$$\frac{E}{K} + \frac{E'}{K'} = 1 + \frac{\pi}{2KK'}$$

### 4.2 Decomposition of the Elliptic Functions

Just as it was possible to decompose the logarithm of an elliptic function into logarithms of theta functions and consequently the function into a product of theta functions, the poles of an elliptic function may be approximated by those of a zeta function for definition of the elliptic function by the sum of its singularities.

Let us first assume that the elliptic function has only simple poles. It may be represented by

$$F(u) = \frac{\theta\left(\frac{u - a_1}{4K}\right) \times \ldots \times \theta\left(\frac{u - a_p}{4K}\right)}{\theta\left(\frac{u - b_1}{4K}\right) \times \ldots \times \theta\left(\frac{u - b_p}{4K}\right)} \times C$$

where all poles $b_1, \ldots b_p$ are distinct.

In the neighborhood of a pole $b_\lambda$, the function behaves like

$$F(u) \sim \frac{\theta\left(\frac{b_\lambda - a_1}{4K}\right) \times \ldots \theta\left(\frac{b_\lambda - a_p}{4K}\right)}{\theta\left(\frac{b_\lambda - b_1}{4K}\right) \times \ldots \theta\left(\frac{b_\lambda - b_p}{4K}\right)} \frac{C}{\cdot\theta\left(\frac{u - b_\lambda}{4K}\right)}$$

The same is true in the neighborhood of a pole which differs from $b_\lambda$ only by multiples of $4K$ and of $4iK'$.

On the other hand, in the neighborhood of the same poles

$$\ln\left[\theta\left(\frac{u - b_\lambda}{4K}\right)\right] \sim \ln\left(\frac{u - b_\lambda}{4K}\right) + \ln\left[\theta'(o)\right]$$

$$Z\left[\frac{u - b_\lambda}{4K}\right] \sim \frac{4K}{u - b_\lambda} \sim \frac{\theta'(o)}{\theta\left(\frac{u - b_\lambda}{4K}\right)}$$

Consequently, the function

$$G(u) = F(u) - C\sum_{\lambda=1}^{\lambda=p} \frac{\theta\left(\frac{b_\lambda - a_1}{4K}\right) \times \ldots \theta\left(\frac{b_\lambda - a_p}{4K}\right)}{\theta'(o) \times \ldots \theta\left(\frac{b_\lambda - b_p}{4K}\right)} Z\left(\frac{u - b_\lambda}{4K}\right)$$

has no singularity at a finite distance. Its derivative is an elliptic function without singularity, even at infinity, by reason of its double periodicity. It can only be a constant and consequently

$$G(u) = A + Bu$$

but $G(u)$ admits the period $4K$ and hence $B$ is zero

$$F(u) = A + C\sum_{\lambda=1}^{\lambda=p} \frac{\theta\left(\frac{b_\lambda - a_1}{4K}\right) \times \ldots \theta\left(\frac{b_\lambda - a_p}{4K}\right)}{\theta'(o) \times \ldots \theta\left(\frac{b_\lambda - b_p}{4K}\right)} Z\left(\frac{u - b_\lambda}{4K}\right)$$

The elliptic function is thus found to be decomposed into zeta functions.

Let us note besides that $F(u)$ admits the period $4iK'$ and that, when $u$ increases by $4iK$, the function $Z\left(\frac{u - b_\lambda}{4K}\right)$ decreases by $2i\pi$. Consequently

$$\sum_{\lambda=1}^{\lambda=p} \frac{\theta\left(\frac{b_\lambda - a_1}{4K}\right) \times . . . \times \theta\left(\frac{b_\lambda - a_p}{4K}\right)}{\theta'(o) \times . . . \times \theta\left(\frac{b_\lambda - b_p}{4K}\right)} = 0$$

This indicates that the sum of the residues of an elliptic function in its poles is zero and expresses simultaneously for a $p$ that is at least equal to 3, a general property of the theta functions which one must not apply without observing that, for the chosen form of the elliptic function, the sum of the complex variables of the zeros is equal to the sum of the complex variables of the poles.

If some poles of the elliptic function are multiples, the calculation is appreciably more complicated.

$$F(u) = C \frac{\theta\left(\frac{u - a_1}{4K}\right) \times . . . \times \theta\left(\frac{u - a_p}{4K}\right)}{\theta^{\alpha}\left(\frac{u - b_\lambda}{4K}\right) \times . . . \times \theta\left(\frac{u - b_p}{4K}\right)}$$

Noting that $\frac{d^{\alpha-1}}{dv^{\alpha-1}} Z(v)$ behaves in the neighborhood of $v = 0$ like $\frac{-(-1)^{\alpha}\,\overline{\alpha - 1!}}{v\alpha}$, we shall form, for $v = \frac{u - b_\lambda}{4K}$, the elliptic function

$$F_1(u) = \frac{F(u)}{C} + (-1)^{\alpha} \frac{d^{\alpha-1} Z(v)}{dv^{\alpha-1}\ \alpha - 1!} \frac{\theta\left(\frac{b_\lambda - a_1}{4K}\right) \times . . . \times \theta\left(\frac{b_\lambda - a_p}{4K}\right)}{\theta'^{\alpha}(o) \times . . . \times \theta\left(\frac{b_\lambda - b_p}{4K}\right)}$$

This new elliptic function admits the pole $u = b_\lambda$ at most to the order $\alpha - 1$; starting again, we shall progressively reduce the order and we shall find the principal part in the neighborhood of each pole. The function $F(u)$ differs from the sum of the principal parts in the neighborhood of each of its poles only by a constant. The elliptic function is thus decomposed into a sum of zeta functions and into derivatives of zeta functions.

For practical purposes, to calculate the principal part in the neighborhood of a multiple pole, one will proceed as for the decomposition of rational fractions, replacing every term by a limited expansion.

### 4.3 Calculation of the Elliptic Integrals

An elliptic integral is defined by $\int P\left[y, \sqrt{(y - y_1)(y - y_2)}, \sqrt{(y - y_3)(y - y_4)}\right] dy$ where $P$ is a rational function of the three variables.

The form of an elliptic integral is invariant in the course of an inversion or of an inversion followed by a symmetry which can be defined by

$$y = \frac{ax + b}{cx + d}$$

The critical points $x_1$, $x_2$, $x_3$, $x_4$ are the images, by inversion symmetry, of the critical points $y_1$, $y_2$, $y_3$, $y_4$.

Let us first assume that the four critical points of the y-plane are not on the same circle, that $C_1$ is the circle which passes through the points $y_2$, $y_3$, $y_4$, that $C_2$ is the circle which passes through $y_3$, $y_4$, $y_1$, etc., (fig. 38). Let us draw the circle $C$ which bisects $C_3$, $C_4$, and the circle $C'$ which bisects $C_1$, $C_2$ and let us make an inversion with respect to one of the points common to these circles. The circles $C$, $C'$ are transformed into straight lines. The circles $C_1$, $C_2$ on one hand, $C_3$, $C_4$ on the other, are transformed into circles of the same radius and, if the origin is chosen at the intersection of the images of $C$, $C'$, if the $x$ axis is chosen in the direction of the two most distant images, for instance $x_1$, $x_2$, if the scale is chosen so that the distance will be 2, the critical points are

$$x_2 = -x_1 = 1 \qquad x_3 = -x_4 = \frac{1}{k}$$

where $k$ is a number which is generally complex and has a modulus smaller than 1.

The elliptic integral takes the form

$$\int P_1\left[x, \sqrt{1 - x^2}, \sqrt{1 - k^2x^2}\right] dx$$

For the method of construction which was followed, $k$ cannot be real, but it can be purely imaginary and in this case the integral is reduced to that of an elliptic function by the transformation

$$x = \mathrm{cn}\left[u, \sqrt{\frac{-k^2}{1 - k^2}}\right]$$

If $k$ is not purely imaginary, the elliptic integral is reduced to that of an elliptic function with imaginary modulus, which exceeds the scope of the present study.

Let us now assume that the critical points in the y-plane are on the same circle $C$, that $y_1$ and $y_2$ are consecutive, and that $C_1$ and $C_2$ are the circles orthogonal to $C$ at $y_1$, $y_3$ on one hand, at $y_2$ and $y_4$ on the other (fig. 39). Finally, let $C_3$ be the circle tangent to $C_1$, $C_2$, and to $C$ between $y_3$ and $y_4$.

If we make an inversion with respect to the point of contact of $C$ and $C_3$, the two circles are transformed into two parallel straight lines. The circles $C_1$ and $C_2$ have therefore the same radius, and choosing the axis of the abscissas following the straight line transformed from $C$, the origin in the middle between $x_1$, $x_2$, and the scale in such a manner that $x_1$, $x_2$ is 2, one obtains

$$x_1 = -x_2 = 1 \qquad x_4 = -x_3 = \frac{1}{k} > 1$$

The elliptic integral thus assumes the form

$$\int P_2\left(x, \sqrt{1 - x^2}, \quad \sqrt{1 - k^2x^2}\right)dx$$

and becomes the integral of an elliptic function by the transformation

$$x = \mathrm{sn}(u,k)$$

If now the four critical points $y_1$, $y_2$, $y_3$, $y_4$ are on a circle $C$, the points $y_3$ and $y_4$ and the points $y_1$ and $y_2$ (fig. 40) are on opposite sides.

The above construction will this time furnish the following values for the images of the critical points

$$x_3 = -x_2 = 1 \qquad x_2 = -x_4 = \frac{1}{\alpha^2} < 1$$

and the elliptic integral takes the form

$$\int P_3\left[x, \sqrt{(1 + x)\left(1 + \alpha^2 x\right)}, \sqrt{(1 - x),\left(1 + \alpha^2 x\right)}\right] dx$$

It is reduced to the previous form by the transformation

$$x = \frac{\left(\alpha^2 - 1\right)x_1{}^2 - \left(\alpha^2 + 1\right)}{\left(\alpha^2 - 1\right)x_1{}^2 + \left(\alpha^2 + 1\right)}$$

Finally, an elliptic integral is the integral of an elliptic function. The latter always has a real modulus when the four critical points of the initial integral are on the same circle or on a straight line. It does generally not have a real modulus when the four critical points of the initial integral are not on the same circle. To make it be effectively a function with real modulus, it is necessary that two circles bisecting the four circles passing through three critical points be orthogonal. This is true in particular for the integral of section 1.6 defining the flow inside an equilateral triangle which was reduced to

$$n = \int \frac{dx}{\sqrt{1 - x^3}}$$

the critical points are $1$, $j$, $j^2$, and infinity. A bisecting circle passing through $j$, $j^2$ admits as diameter the degenerated circle connecting $1$ with infinity.

With the elliptic function being decomposed into zeta functions and, if multiple poles exist, into derivatives of zeta functions, the integration is immediate. To the simple poles correspond logarithms of theta functions represented by source vortices. Doublets correspond to the double poles, doublets of a high order to multiple poles.

A first example corresponds to the addition formula of $Z_3$.

$$\int \left[\operatorname{sn}(2Kv)\right]\left[\operatorname{sn} 2K\left(v + v_1\right)\right] 2K\, dv = \frac{1}{k^2 \operatorname{sn}\left(2Kv_1\right)}\left[\ln \frac{\theta_3(v)}{\theta_3\left(v + v_1\right)} + vZ_3\left(v_1\right)\right]$$

Other examples will be given below.

## 4.4 Elliptic Integral of the Second Kind and Flows Around Rectangles

Before transformation into integrals of elliptic functions, the elliptic integrals may be decomposed into integrals of three kinds. The calculations are generally more tedious than in the case of immediate transformation into elliptic functions, but the integrals of three kinds have a direct advantage for many applications. The integral of the first kind is connected with the elliptic functions and was studied in the second chapter. The integrals of the second and of the third kind will be studied.

The elliptic integral of the second kind is defined by

$$el = \int_0^x \sqrt{\frac{1 - k^2x^2}{1 - x^2}}\, dx$$

According to the first chapter, it corresponds to the flow in the neighborhood of an open rectangular polygon (fig. 41). The calculation of this integral is immediate. It is sufficient to put

$$x = \operatorname{sn}(u,k)$$

for finding

$$el(u,k) = \int_0^u \operatorname{dn}^2(u,k)du$$

The vertices at acute angles toward the flow correspond to $u = \pm K$. For $u = K$

$$x = \operatorname{sn}(K,k) = 1$$

$$el(K,k) = \int_0^K \operatorname{dn}^2(u,k)du = E(k)$$

a value already introduced in 4.1, in the study of the derivative of the zeta function.

The vertices at obtuse angles toward the flow correspond to

$$u = \pm K + iK'$$

For $u = K + iK'$

$$x = \mathrm{sn}(K + iK') = \frac{1}{k}$$

$$e l(K + iK',k) = E(k) + \int_K^{K+iK'} \mathrm{dn}^2(u,k)du$$

or, making the change of variable:

$$u = K + iK' + iu'$$

$$\mathrm{dn}(K + iK' + iu',k) = -k' \ \mathrm{sn}(u',k')$$

$$\mathrm{dn}^2(K + iK' + iu',k) = 1 - \mathrm{dn}^2(u',k')$$

Consequently

$$e l(K + iK',k) = E(k) + i\left[K'(k') - E'(k')\right]$$

The function $e l(u)$ is linked to the $Z_3$ function

$$e l(u,k) = \frac{1}{2K} Z_3\left(\frac{u}{2K}\right) + \frac{Eu}{K}$$

One finds again that for $u = K$, when $Z_3 = 0$, the value of $e l(u)$ is $E$, and that for $u = K + iK'$, when $Z_3 = -i\pi$, the value of $e l(u)$ is

$$e l(K + iK',k) = \frac{-i\pi}{2K} + E + \frac{iK'E}{K} = E + i(K' - E')$$

In the plane of the variable $u$, the field of $e l(u)$ differs from the field of $Z_3(u)$ only by superposition of a uniform stream which displaces the stagnation points to lead them to the zeros of $\mathrm{dn}\ u$ (fig. 42).

Of more interest for the applications is the integral

$$\int_0^x \sqrt{\frac{1 - x^2}{1 - k^2x^2}}\, dx$$

whose representative field (fig. 43) is, according to the first chapter, that of a flow around a rectangle.

It is easy to normalize (fig. 44), by setting

$$k = \sin X_o \qquad x = \frac{\sin X}{\sin X_o}$$

whence follows the new form of the integral

$$\int \frac{\sqrt{\sin^2 X_o - \sin^2 X}}{\sin X_o}\, dX$$

but it is more practical to reestablish the connection with the elliptic functions by setting $x = \mathrm{sn}(u,k)$, and this transformation gives to the integral the form

$$\int_o^u \mathrm{cn}^2 du = \int_o^u \frac{\mathrm{dn}^2 u - k'^2}{k^2}\, du$$

$$= \frac{1}{k^2}\, el(u) - \frac{k'^2}{k^2}\, u$$

$$= \frac{1}{2Kk^2}\, Z_3\left(\frac{u}{2K}\right) + \left(\frac{E}{K} - k'^2\right)\frac{u}{k^2}$$

One of the vertices corresponds to $u = K$ where $Z_3 = 0$ and the value of the integral is $\frac{E - k'^2 K}{k^2}$. The middle of the side adjacent to the origin corresponds to $u = K + iK'$ where $Z_3 = -i\pi$ and the value of the integral is

$$-\frac{i\pi}{2Kk^2} + \left(\frac{E}{K} - k'^2\right)\frac{K + iK'}{k^2} = \frac{E - k'^2 K}{k^2} - i\,\frac{E' - k^2 K'}{k^2}$$

In the plane of the variable $u$, the field of the new integral again differs from that of $Z_3$ only by a uniform stream (fig. 45), this time leading the stagnation points to the vertices of the rectangle which correspond to the zeros of $\mathrm{cn}\, u$.

It is equally interesting to construct the field of $\mathrm{sn}\, u$ in the plane of the variable represented by the integral

$$Z_3\left(\frac{u}{2K}, q\right) = 2K\int_o^u \left[\mathrm{dn}^2 u - \frac{E}{K}\right] du$$

This field is represented by figure 46.

In a general manner, the field of $\operatorname{sn} u$ in the plane of the elliptic integral of the second kind, increased by a quantity proportional to $u$, is uniform at infinity and surrounds a rectangular polygon the forms of which vary with the real proportionality factor.

### 4.5 Elliptic Integral of the Third Kind

The elliptic integral of the third kind is defined by

$$\int_0^x \frac{1}{x - \alpha} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

In contrast to the two preceding integrals, it depends on two parameters: $k$ and $\alpha$.

Taking into consideration the methods of integration applied above, it is useful to study the integrals of the third kind only in the cases of direct significance for the applications. These latter correspond almost always to real values of $\alpha$ which alone are considered here.

The position of $\alpha$ with respect to the critical points: $-1/k$, $-1$, $1$, $1/k$ determines the appearance of the rectangular polygon, corresponding to the real values of $\alpha$ and limiting the representative field which, in accordance with the first chapter, is that of a doublet of the order $1/2$.

Figure 47(a) corresponds to a value of $\alpha$ lying between $-1$ and $1$.

Figure 47(b) corresponds to a value of $\alpha$ between $1$ and $1/k$.

Figure 47(c) corresponds to a value of $\alpha$ larger than $1/k$.

It is of no interest whatever to normalize here in defining the field of a vortex. It is preferable to set

$$x = \operatorname{sn}(u,k) \qquad \alpha = \operatorname{sn}(u_0,k)$$

This transformation leads to the integral

$$\int_0^u \frac{du}{\operatorname{sn} u - \operatorname{sn} u_0}$$

which can be easily represented in the plane of the variable $u$.

For $u_o$ between $-K$ and $K$ (fig. 48(a)), the function to be integrated presents a pole at $u = u_o$ in the neighborhood of which it behaves like

$$\frac{1}{\operatorname{sn}\left[u_o + (u - u_o)\right] - \operatorname{sn} u_o} \sim \frac{1}{\operatorname{cn} u_o \operatorname{dn} u_o (u - u_o)}$$

and consequently the integral behaves like $\frac{1}{\operatorname{cn} u_o \operatorname{dn} u_o} \ln(u - u_o)$. It is represented by a source. It is real, like its derivative, for $-K < u < K$ and for $-K + iK' < u < K + iK'$. It is purely imaginary along the vertical lines of abscissas $-K$ and $+K$ since its derivative is real and $du$ is imaginary on these straight lines. The derivative is zero for $u = iK'$ which corresponds to a stagnation point of the stream.

For $u_o$ lying between $K$ and $K + iK'$ (fig. 48(b)), the same analysis defines the field of a vortex since $\operatorname{cn} u_o$ is purely imaginary.

For $u$ lying between $-K + iK'$ and $K + iK'$ (fig. 48(c)), the field is again that of a source since $\operatorname{cn} u_o \operatorname{dn} u_o$ is real.

In order to represent the elliptic function with the aid of zeta functions, one must determine all poles in a domain of variation of $u$ from $4K$ to $4iK'$. These poles are the zeros of $\operatorname{sn} u - \operatorname{sn} u_o$:

$$u_o \qquad u_o + 2iK' \qquad -u_o + 2K \qquad -u_o + 2K + 2iK'$$

We have already calculated the principal part in the neighborhood of $u = u_o$

$$\frac{1}{\operatorname{sn} u - \operatorname{sn} u_o} \sim \frac{1}{\operatorname{cn} u_o \operatorname{dn} u_o (u - u_o)} \sim \frac{Z\left(\frac{u - u_o}{4K}\right)}{4K \operatorname{cn} u_o \operatorname{dn} u_o}$$

Proceeding in the same manner in the neighborhood of the three other poles, one has

$$\frac{4K \operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn} u - \operatorname{sn} u_o} = Z\left(\frac{u - u_o}{4K}\right) + Z_3\left(\frac{u - u_o}{4K}\right) - Z_1\left(\frac{u + u_o}{4K}\right) - Z_2\left(\frac{u + u_o}{4K}\right) + C^{te}$$

The constant can be determined for $u = u_o + 2K$

$$\frac{4K \operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn} u - \operatorname{sn} u_o} + \frac{2K \operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn} u_o} = Z\left(\frac{u - u_o}{4K}\right) + Z_3\left(\frac{u - u_o}{4K}\right) - Z_1\left(\frac{u + u_o}{4K}\right) - Z_2\left(\frac{u + u_o}{4K}\right) + Z\left(\frac{u_o}{2K}\right) + Z_3\left(\frac{u_o}{2K}\right)$$

and the integration is immediate

$$\int_0^u \frac{\operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn} u - \operatorname{sn} u_o} du = \ln \frac{\theta\left(\frac{u - u_o}{4K}\right)\theta_3\left(\frac{u - u_o}{4K}\right)}{\theta_1\left(\frac{u - u_o}{4K}\right)\theta_2\left(\frac{u + u_o}{4K}\right)} \frac{\theta_1\left(\frac{u_o}{4K}\right)\theta_2\left(\frac{u_o}{4K}\right)}{\theta\left(\frac{u_o}{4K}\right)\theta_3\left(\frac{u_o}{4K}\right)} + \frac{u}{4K}\left[-\frac{2K \operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn} u_o} + Z\left(\frac{u_o}{2K}\right) + Z_3\left(\frac{u_o}{2K}\right)\right]$$

One reduces the elliptic integral of the third kind to the integrals of:

$$\frac{1}{\operatorname{sn}^2 u - \operatorname{sn}^2 u_o} \qquad \frac{\operatorname{sn} u}{\operatorname{sn}^2 u - \operatorname{sn}^2 u_o}$$

Agreeing to introduce parasitic singularities which compensate one another, one has

$$\frac{1}{\operatorname{sn} u - \operatorname{sn} u_o} = \frac{\operatorname{sn} u + \operatorname{sn} u_o}{\operatorname{sn}^2 u - \operatorname{sn}^2 u_o} = \frac{\operatorname{sn} u}{\operatorname{sn}^2 u - \operatorname{sn}^2 u_o} + \frac{\operatorname{sn} u_o}{\operatorname{sn}^2 u - \operatorname{sn}^2 u_o}$$

We shall proceed inversely, deducing the two new integrals from that which has just been calculated

$$\int_0^u \frac{2\operatorname{sn} u_o \operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn}^2 u - \operatorname{sn}^2 u_o} du = \int_0^u \frac{\operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn} u - \operatorname{sn} u_o} du - \int_0^u \frac{\operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn} u + \operatorname{sn} u_o} du$$

$$= \ln \frac{\theta\left(\frac{u_o - u}{2K}\right)}{\theta\left(\frac{u_o + u}{2K}\right)} + u\left[-\frac{\operatorname{cn} u_o \operatorname{dn} u_o}{\operatorname{sn} u_o} + \frac{1}{2K} Z\left(\frac{u_o}{2K}\right)\right]$$

Likewise

$$\int_0^u \frac{2\text{cn } u_o \text{ dn } u_o \text{ sn } u}{\text{sn}^2 u - \text{sn}^2 u_o} du = \int_0^u \frac{\text{cn } u_o \text{ dn } u_o}{\text{sn } u - \text{sn } u_o} du + \int_0^u \frac{\text{cn } u_o \text{ dn } u_o}{\text{sn } u + \text{sn } u_o} du$$

but it is more convenient, in order to avoid the reduction of the theta functions, to calculate this integral directly by making the change of variable $u = K - u_1$ which leads to a rational integral in $\text{dn } u_1$.

One can likewise observe that the field of the function $\dfrac{1}{\text{sn}^2 u - \text{sn}^2 u_o}$ differs from that of the function $\dfrac{1}{\text{sn } u - \text{sn } u_o}$ only by the ratio of the periods, and one can thus reduce the two integrals to one another by the formulas of doubling of the periods.

### 4.6 Calculation of the Zeta Functions and of Various Constants

The most convenient method of calculation of the zeta functions utilizes the expansions in trigonometric series of the theta functions.

$$Z(v,q) = \frac{\frac{d}{dv}\left[\theta(v,q)\right]}{\theta(v,q)} = \pi \frac{\sum_0^\infty (-1)^r q^{\left(r+\frac{1}{2}\right)^2} (2r+1)\cos(2r+1)\pi v}{\sum_0^\infty (-1)^r q^{\left(r+\frac{1}{2}\right)^2} \sin(2r+1)\pi v}$$

and the analogous formulas for $Z_1$, $Z_2$, $Z_3$.

It can be equally convenient to directly make use of the expansions in trigonometric series. It is sufficient to start from expansions of the theta functions into infinite products

$$\ln \theta_3(v,q) = \ln \theta_3(o,q) - 2\ln\left[\prod_0^\infty \left(1 - q^{2r+1}\right)\right] + \sum_{r=0}^\infty \ln\left(1 - q^{2r+1}e^{2i\pi v}\right) +$$

$$\sum_{r=0}^\infty \ln\left(1 - q^{2r+1}e^{-2i\pi v}\right)$$

$$Z_3(v,q) = -2\pi i \sum_{r=0}^{\infty} \left[\frac{q^{2r+1}\, e^{2i\pi v}}{1 - q^{2r+1}\, e^{2i\pi v}}\right] + 2\pi i \sum_{r=0}^{\infty}\left[\frac{q^{2r+1}\, e^{-2i\pi v}}{1 - q^{2r+1}\, e^{-2i\pi v}}\right]$$

In expanding, at least for the real values of $v$, such that $q^{2r+1}\, e^{\pm 2i\pi v}$ has a modulus smaller than one:

$$Z_3(v,q) = -2\pi i \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} q^{(2r+1)s} e^{2is\pi v} + 2\pi i \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} q^{(2r+1)s}\, e^{-2is\pi v}$$

$$= 4\pi \sum_{s=1}^{\infty} \frac{q^s}{1 - q^{2s}} \sin(2s\pi v)$$

Likewise

$$Z_2(v,q) = Z_3\left(v + \frac{1}{2}\right) = 4\pi \sum_{s=1}^{\infty} (-1)^s \frac{q^s}{1 - q^{2s}} \sin(2s\pi v)$$

For $Z$ and $Z_1$ one must isolate the poles

$$Z(v,q) = \pi \cot(\pi v) + 4\pi \sum_{s=1}^{\infty} \frac{q^{2s}}{1 - q^{2s}} \sin(2s\pi v)$$

$$Z_1(v,q) = -\pi \tan(\pi v) + 4\pi \sum_{s=1}^{\infty} (-1)^s \frac{q^{2s}}{1 - q^{2s}} \sin(2s\pi v)$$

The expansions of the derivatives of the zeta functions are deduced from the expansions above, and for instance

$$\frac{dZ_3}{dv} = 4K^2\left[\mathrm{dn}^2 2Kv - \frac{E}{K}\right] = 8\pi^2 \sum_{s=1}^{\infty} \frac{sq^s}{1 - q^{2s}} \cos(2s\pi v)$$

This defines $E(q)$ for $v = 0$

$$\frac{K[K - E]}{2\pi^2} = \sum_{s=1}^{\infty} \frac{sq^s}{1 - q^{2s}}$$

One may also utilize, for the calculation of $E$, the theta functions

$$Z_3 = \frac{1}{\theta_3}\frac{d\theta_3}{dv}$$

$$\frac{dZ_3}{dv} = \frac{1}{\theta_3}\frac{d^2\theta_3}{dv^2} - \frac{1}{\theta_3^2}\left(\frac{d\theta_3}{dv}\right)^2$$

and for $v = 0$

$$\frac{dZ_3}{dv} = \frac{8\pi^2 \sum_{n=1}^{\infty} (-1)^{n+1} n^2 q^{n^2}}{1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}}$$

$$\frac{K(K - E)}{2\pi^2} = \frac{\sum_{n=1}^{\infty} (-1)^{n+1} n^2 q^{n^2}}{1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}}$$

The decomposition of the elliptic functions into zeta functions introduces

$$\theta'(o,q) = \frac{d}{dv}\Big[\theta(v,q)\Big]_{v=0}$$

This quantity may be calculated in various ways, for instance starting from the expression of $\operatorname{sn} u$

$$\operatorname{sn} u = \frac{\theta\left(\frac{u}{2K}\right)\theta_2(o)}{\theta_3\left(\frac{u}{2K}\right)\theta_1(o)}$$

Deriving

$$\operatorname{cn} u \operatorname{dn} u = \frac{1}{2K}\left[\frac{\theta_1\left(\frac{u}{2K}\right)}{\theta_3\left(\frac{u}{2K}\right)} - \frac{\theta_3'\left(\frac{u}{2K}\right)\theta\left(\frac{u}{2K}\right)}{\theta_3^2\left(\frac{u}{2K}\right)}\right]\frac{\theta_2(o)}{\theta_1(o)}$$

and for $u = 0$

$$1 = \frac{1}{2K} \frac{\theta'(o)}{\theta_3(o)} \frac{\theta_2(o)}{\theta_1(o)}$$

taking into account

$$\frac{2K}{\pi} = \theta_2{}^2(o)$$

we shall find

$$\theta'(o) = \pi\theta_1(o)\theta_2(o)\theta_3(o)$$

This formula specifies the expansion of the theta function into an infinite product

$$\theta(v,q) = \theta_1(o)\theta_2(o)\theta_3(o)\sin \pi v \prod_{r=1}^{\infty} \left[\frac{\left(1 - q^{2r}e^{2i\pi v}\right)\left(1 - q^{2r}e^{-2i\pi v}\right)}{\left(1 - q^{2r}\right)^2}\right]$$

Finally, the principal constants may be calculated with the aid of the following formulas

$$\sqrt{\frac{2K}{\pi}} = \theta_2(o) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$

$$\sqrt{k} = \sqrt{\sin \Theta} = \frac{\theta_1(o)}{\theta_2(o)} = \frac{2\sum\limits_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}}{1 + 2\sum\limits_{n=1}^{\infty} q^{n^2}}$$

$$\sqrt{k'} = \sqrt{\cos \Theta} = \frac{\theta_3(o)}{\theta_2(o)} = \frac{1 + 2\sum\limits_{n=1}^{\infty} (-1)^n q^{n^2}}{1 + 2\sum\limits_{n=1}^{\infty} q^{n^2}}$$

$$1 - \frac{E}{K} = \frac{2\pi^2}{K^2} \frac{\theta_3''(o)}{\theta_3(o)} = \frac{1}{2\theta_2{}^4(o)} \frac{\theta_3''(o)}{\theta_3(o)} = \frac{1}{2} \frac{1}{\left(1 + 2\sum\limits_{n=1}^{\infty} q^{n^2}\right)^4} \frac{\sum\limits_{n=1}^{\infty} (-1)^{n+1} n^2 q^{n^2}}{1 + \sum\limits_{n=1}^{\infty} (-1)^n q^{n^2}}$$

It will always be possible to assume in these formulas $K' \geqslant K$ and $q \leqslant e^{-\pi}$ by interchanging eventually the roles of $K$ and $K'$.

Moreover, the infinite products of the expressions of the theta functions may be reduced to series.

For instance

$$\ln\left[\prod_{r=1}^{\infty}\left(1 - q^{2r}\right)\right] = \sum_{r=1}^{\infty} \ln\left(1 - q^{2r}\right)$$

$$= -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{1}{s} q^{2rs}$$

$$= -\sum_{s=1}^{\infty} \frac{1}{s} \frac{q^{2s}}{1 - q^{2s}}$$

## 4.7 Second Integrals of Elliptic Functions

If one must again integrate the integral of an elliptic function, the weighted sum of logarithms of theta functions and of zeta functions or of derivatives of these latter, only the logarithms introduce new functions.

For the applications and especially for the numerical calculations it may be sufficient to define these new functions by series. For instance

$$\ln\left[\theta_3(v)\right] = \ln\theta_3(o) + \sum_{r=0}^{\infty} \ln\left[\frac{\left(1 - q^{2r+1}e^{2i\pi v}\right)\left(1 - q^{2r+1}e^{-2i\pi}\right)}{\left(1 - q^{2r+1}\right)^2}\right]$$

$$= \ln\theta_3(o) + \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} \frac{q^{(2r+1)s}}{s}\left[2e^{2si\pi v}e^{-2si\pi v}\right]$$

$$= \ln\theta_3(o) + 4\sum_{s=1}^{\infty} \frac{q^s}{s\left(1 - q^{2s}\right)} \sin^2 s\pi v$$

$$\int\left[\ln\theta_3(v)\right]dv = v\ln\left[\theta_3(o)\right] + 2\sum_{s=1}^{\infty} \frac{q^s}{s\left(1 - q^{2s}\right)}\left(v - \frac{\sin 2s\pi v}{2s\pi}\right)$$

In a general manner, the complicated integrals pertaining to elliptic functions may be calculated numerically by use of expansions in series. The method does not make evident the singularities of the functions except those which are situated on the contour of integration; they are most frequently led back to the real axis and must be removed from the expansions in series.

## 5. VARIOUS APPLICATIONS

### 5.0 Ranges of Application

The present paper was written so as to permit understanding and facilitate utilization of the studies of fluid mechanics for which we introduced elliptic functions. We refer to these studies (1 to 5) which form a group of applications.

The elliptic functions solve the problems of flows around obstacles the contours of which are schematized by segments of straight lines and of stream lines. They furnish the field of nonstationary flow around plane vanes in vibration.

The examples chosen below illustrate the possibilities of use.

### 5.1 Laminar Profiles for Airplane Wings and Blade Grids

The problem is to define profiles along which the velocity varies very slowly in order to avoid separation of the boundary layer and to delay the appearance of shock waves at high speeds. The importance of this problem is emphasized in several of our publications and in the work done by my coworkers under my direction (6 to 15).

The airplane wing is a particular case of a blade grid for an infinite blade pitch. It is therefore sufficient to study the blade grid.

For the first schematization which will be corrected later on, we assume that the velocity is rigorously constant on certain regions of the profiles. The pressure is then equally constant for the isentropic potential flow, and the contour is that of a free jet in equilibrium with an inert space at constant pressure. In order to simplify the calculations, the connections between the above regions are assumed to be rectilinear.

A profile may thus be defined by a dihedron AB, AB' at the leading edge, two stream lines BC and B'C', two segments CD, with C'D forming a dihedron at the trailing edge (fig. 49). The lower surface of the profile, convex near the leading edge and concave in the central region if the camber is great, normally admits a point of inflection I.

We disregard the possibilities of application to fluids satisfying a judiciously chosen law of compressibility which result from a simple modification of the integration beginning with the hodograph, and treat only the flow of the incompressible fluid (3 and 10 to 15).

If $F$ is the complex potential of the flow in the physical plane where the complex variable is $z$, we construct the hodograph $\zeta(F) = dz/dF$ or rather the logarithmic hodograph $\lambda = \ln\zeta$.

Along the stream lines, the intensity of the velocity is constant, the modulus of $\zeta$ and the real part of $\lambda$ are constants.

Along the segments of straight lines, the direction of the velocity is constant, the argument of $\zeta$ and the imaginary part of $\lambda$ are constants.

The image of the profile in the plane of the hodograph is therefore a rectangular polygon (fig. 50), the general appearance of which can be easily constructed from an approximate outline of the flow.

The field $F(\lambda)$ is that of a source vortex with the image $O$ of the infinity upstream discharging into a sink vortex with the image $O'$ of the infinity downstream. In the case of a wing profile, this field is reduced to the field of a doublet with circulation with the unique image of infinity. It is not difficult to define as in 1.7, if one knows how to make a half plane correspond to the interior of the rectangular polygon by a conformal transformation.

Let us choose to define such a conformal transformation $f(\lambda)$ by the field of a doublet at $N$ on $C'D$ (fig. 51). The reduction of the elliptic integrals shows that it is always possible to choose the point $N$, the origin and the scale of the field in such a manner that the function assumes the values $-1$ and $1$ at $B$ and $B'$, $1/k$ and $-1/k$ at $C'$ and $C$.

Let us thus assume $f_1$, $f_2$, $f_3$ to be the values of $f$ at $A$, the image of the stagnation point, $D$ the image of the trailing edge, $I$ the image of the point of inflection at the lower surface of the profile; the derivative $d\lambda/df$ then is determined by its singularities except for one factor:

$$\frac{d\lambda}{df} = \frac{C^{te}}{\sqrt{1 - f^2}}\sqrt{\frac{1 + kf}{1 - kf}}\,\frac{f - f_3}{(f - f_1)(f - f_2)}$$

According to figure 41, the constants satisfy the inequalities

$$-1 < f_1 < 1$$

$$f_2 < -1/k$$

$$-1/k < f_3 < -1$$

and if one sets

$$f = \operatorname{sn} u \qquad f_1 = \operatorname{sn} u_1 \qquad f_2 = \operatorname{sn}(u_2 + iK')$$

the function $d\lambda/du$ is elliptic:

$$\frac{d\lambda}{du} = -C_o \frac{(1 + k \operatorname{sn} u)(\operatorname{sn} u - f_3)}{(\operatorname{sn} u - \operatorname{sn} u_1)\left[\operatorname{sn} u - \operatorname{sn}(u_2 + iK')\right]}$$

The constant $C_o$ is real and positive because $u$ is real along BA, and contained between $-K$ and $u_1$, whereas $d\lambda/du$ is real and positive.

It is convenient here to decompose the function which depends only on $\operatorname{sn} u$,

$$-\frac{1}{C_o}\frac{d\lambda}{du} = +k + \frac{\alpha_1}{\operatorname{sn} u - \operatorname{sn} u_1} + \frac{\alpha_2}{\operatorname{sn} u - \operatorname{sn}(u_2 + iK')}$$

by replacing the parameter $f_3$ by the parameters $\alpha_1$ and $\alpha_2$ linked by the condition that $d\lambda/du$ should be zero for $k \operatorname{sn} u = -1$

$$1 = \frac{\alpha_1}{1 + k \operatorname{sn} u_1} + \frac{\alpha_2}{1 + k \operatorname{sn}(u_2 + iK')} = \frac{\alpha_1}{1 + k \operatorname{sn} u_1} + \frac{\alpha_2 \operatorname{sn} u_2}{1 + \operatorname{sn} u_2}$$

The function $\lambda(u)$ is the sum of integrals of the third kind and of a linear function of $u$.

Before proceeding with the calculation, it is suitable to remark that extensive work would not be justified for a crude representation. A much more significant profile may be obtained for a continuous contour of a hodograph and close to the polygonal contour. It is possible to retain two stream lines of the field of a source at A discharging into a sink at D (fig. 52). The two branches at infinity in A, the image of the stagnation point, must be separated from $i\pi$ so that the profile does not have an angular point at the leading edge. The two straight lines BA, B'A of the polygonal contour must therefore be separated $i(\pi + \epsilon + \epsilon')$ where $\epsilon$ and $\epsilon'$ designate arbitrary small angles fixing the deviation between the hodograph of the schematic profile and that of the desired continuous profile, and presenting, in particular, a finite curvature at the leading edge.

In the neighborhood of the point A, the derivative $d\lambda/du$ behaves like

$$\frac{d\lambda}{du} \sim \frac{-\alpha_1 C_o}{\text{sn } u - \text{sn } u_1} \sim \frac{\alpha_1 C_o}{\text{cn } u_1 \text{ dn } u_1} \frac{1}{u - u_1}$$

and the integral, except for a regular function, like

$$\lambda \sim \frac{-\alpha_1 C_o}{\text{cn } u_1 \text{ dn } u_1} \ln(u - u_1)$$

When $u$ exceeds the value $u_1$ by increasing values, the imaginary part of $\ln(u - u_1)$ decreases by $i\pi$ and that of $\lambda$ increases by $i(\pi + \epsilon + \epsilon')$; consequently

$$\alpha_1 C_o = \left(1 + \frac{\epsilon + \epsilon'}{\pi}\right) \text{cn } u_1 \text{ dn } u_1$$

Likewise, in the neighborhood of the point D, the derivative $d\lambda/du$ behaves like

$$\frac{d\lambda}{du} \sim \frac{-\alpha_2 C_o}{\text{sn } u - \text{sn}(u + iK')} \sim \frac{k \text{ sn}^2 u_2}{\text{cn } u_2 \text{ dn } u_2} \frac{\alpha_2 C_o}{u - u_2 - iK'}$$

and the integral, except for a regular function, like

$$\lambda \sim \frac{+\alpha_2 C_o k \text{ sn}^2 u_2}{\text{cn } u_2 \text{ dn } u_2} \ln\left[u - u_2 - iK'\right]$$

When $u$ exceeds the value $u_2 + iK'$ by increasing real variations, $\ln(u - u_2 - iK')$ increases by $+i\pi$, and if $\Delta$ denotes the edge angle of the leading edge, $\lambda$ varies by $-\left(1 + \frac{\epsilon + \epsilon'}{\pi}\right) i\Delta$

$$\alpha_2 C_o = -\left(1 + \frac{\epsilon + \epsilon'}{\pi}\right) \frac{\Delta}{\pi} \frac{\text{cn } u_2 \text{ dn } u_2}{k \text{ sn}^2 u_2}$$

According to the relationship between $\alpha_1$ and $\alpha_2$ established before

$$\frac{C_o}{1 + \frac{\epsilon + \epsilon'}{\pi}} = \frac{\text{cn } u_1 \text{ dn } u_1}{1 + k \text{ sn } u_1} - \frac{\Delta}{\pi} \frac{\text{cn } u_2 \text{ dn } u_2}{k\left(1 + \text{sn } u_2\right) \text{sn } u_2} = C_1$$

Hence the expression of the derivative

$$\frac{1}{1+\frac{\epsilon+\epsilon'}{\pi}}\frac{d\lambda}{du} = -kC_1 - \frac{\mathrm{cn}\,u_1\,\mathrm{dn}\,u_1}{\mathrm{sn}\,u - \mathrm{sn}\,u_1} - \frac{\Delta}{\pi}\frac{\mathrm{cn}(u_2+iK')\,\mathrm{dn}(u_2+iK')}{\mathrm{sn}\,u - \mathrm{sn}(u_2+iK')}$$

Replacing $C_1$ by its expression and integrating beginning with the value $\lambda_B$ of $\lambda$ at B where $u = -K$, one obtains

$$\frac{\lambda-\lambda_B}{1+\frac{\epsilon+\epsilon'}{\pi}} = -\ln\left[-\frac{\theta\left(\frac{u-u_1}{4K}\right)\theta_3\left(\frac{u-u_1}{4K}\right)}{\theta_1\left(\frac{u+u_1}{4K}\right)\theta_2\left(\frac{u+u_1}{4K}\right)}\right] -$$

$$\frac{\Delta}{\pi}\ln\left[-\frac{\theta\left(\frac{u-u_2-iK'}{4K}\right)\theta_3\left(\frac{u-u_2-iK'}{4K}\right)}{\theta_1\left(\frac{u+u_2+iK'}{4K}\right)\theta_2\left(\frac{u+u_2+iK'}{4K}\right)}e^{-i\pi\frac{u+K}{2K}}\right] +$$

$$\frac{u+K}{4K}\left[\frac{2K\,\mathrm{cn}\,u_1\,\mathrm{dn}\,u_1}{\mathrm{sn}\,u_1}\times\frac{1-k\,\mathrm{sn}\,u_1}{1+k\,\mathrm{sn}\,u_1} - Z\left(\frac{u_1}{2K}\right) - Z_3\left(\frac{u_1}{2K}\right)\right] +$$

$$\frac{\Delta}{\pi}\frac{u+K}{4K}\left[\frac{2K\,\mathrm{cn}\,u_2\,\mathrm{dn}\,u_2}{\mathrm{sn}\,u_2}\times\frac{1-\mathrm{sn}\,u_2}{1+\mathrm{sn}\,u_2} - Z\left(\frac{u_2}{2K}\right) - Z_3\left(\frac{u_2}{2K}\right)\right]$$

The second term is expressed with the aid of theta functions that can be calculated for values of the variable which are complex when u varies from -K to K by real values. It is easy to obtain an expression which can be calculated more easily by first coming back to a single theta function

$$-\frac{\Delta}{\pi}\ln\left[\frac{\theta\left(\frac{u-u_2-iK'}{4K}\right)\theta\left(\frac{u-u_2+iK'}{4K}\right)}{\theta\left(\frac{u+u_2+2K+iK'}{4K}\right)\theta\left(\frac{u+u_2+2K-iK'}{4K}\right)}\right]$$

and then decomposing the products of theta functions according to the formula of section 3.4

$$-\frac{\Delta}{\pi}\ln\left[\frac{\theta^2\left(\frac{u-u_2}{4K}\right)}{\theta_1^{\,2}\left(\frac{u+u_2}{4K}\right)}\;\frac{k+\frac{1+\mathrm{dn}(u-u_2)}{1-\mathrm{cn}(u-u_2)}}{k+\frac{1+\mathrm{dn}(u+u_2)}{1+\mathrm{cn}(u+u_2)}}\right]$$

The point $B'$ corresponds to $u = K$

$$\frac{\lambda_{B'} - \lambda_B}{1 + \frac{\epsilon + \epsilon'}{\pi}} = i\pi + K \frac{\mathrm{cn}\, u_1\, \mathrm{dn}\, u_1}{\mathrm{sn}\, u_1} \frac{1 - k\, \mathrm{sn}\, u_1}{1 + k\, \mathrm{sn}\, u_1} \frac{1}{2} Z\left(\frac{u_1}{2K}\right) - \frac{1}{2} Z_3\left(\frac{u_1}{2K}\right) +$$

$$\frac{\Delta}{\pi} K \frac{\mathrm{cn}\, u_2\, \mathrm{dn}\, u_2}{\mathrm{sn}\, u_2} \frac{1 - \mathrm{sn}\, u_2}{1 + \mathrm{sn}\, u_2} - \frac{\Delta}{2\pi} Z\left(\frac{u_2}{2K}\right) - \frac{\Delta}{2\pi} Z_3\left(\frac{u_2}{2K}\right)$$

The real part of $\lambda_{B'} - \lambda_B$ defines — approximately for the modified representation — the logarithm of the ratio of the maximum velocities on the upper and lower surface of the profile.

The calculation of $\lambda(u)$ along the straight lines CA, C'A of the hodograph is comparable. The variable $u$ varies by real values from $-K + iK'$ to $K + iK'$, and this time it is convenient to make the transformation of the theta functions containing $u_1$.

The calculations along the straight lines BC and B'C' introduce the theta functions for complex values of the variable into the two logarithms. It suffices to use the formulas of change of given axes at the end of 3.1 for arriving at a formula analogous to those used above.

To plot the outline of the modified contour of the hodograph, one must use necessarily the theta functions for complex values of the variable, but since the values of $\epsilon$ and $\epsilon'$ are small, the calculations converge rather rapidly.

One will begin with defining the image of the field of figure 52 in the plane of the variable $u$ (fig. 53).

This image $g(u)$ is that of a source at $u_1$ discharging into a sink at $iK' + u_2$ with a field which follows the contour of the rectangle $-K$, $K$, $K + iK'$, $-K + iK'$.

The function $g(u)$ is determined by the above singularities.

$$g(u) = \ln\left[\frac{\mathrm{sn}\, u - \mathrm{sn}\, u_1}{\mathrm{sn}\, u - \mathrm{sn}\left(u_2 + iK'\right)}\right]$$

$$= \ln\left[\frac{\mathrm{sn}\, u - \mathrm{sn}\, u_1}{\mathrm{sn}\, u - \frac{1}{k\, \mathrm{sn}\, u_2}}\right]$$

If the field of the flow around the profile is not desired, it is not necessary to calculate $\lambda(u)$ in the entire field of the hodograph. The contour suffices for the integration of the profile. However, the singularities must be correctly placed. For practical purposes, one must start from approximations furnished by a freehand sketch or by electrical analogy and then make corrections with the aid of the exact formulas. It is, besides, possible to differentiate the latter for corrections in order to avoid calculation of the theta functions.

We do not take up again the calculation of the integration of the profile beginning with the hodograph which is set forth in several of the papers cited as references. However, let us note that this calculation leads to the use of the functions $e^{\lambda}$ and $e^{-\lambda}$ and that the logarithms of the theta functions which appear in $\lambda$ are transformed into powers of theta functions.

## 5.2 Vibrations of a Swallow-Tail Wing

We have studied the flow around a delta wing in vibration (ref. 16) and we intend here to extend the results to a plane, slender swallow-tail wing (fig. 54).

We shall not resume the discussion of the approximations which permit definition of the flow potential by the real part of

$$\Phi(\zeta,x,\tau) = \int W_n\left(y_o,x,\tau\right)f\left(\zeta,y_o\right)dy_o$$

where $x$ is the abscissa counted from the vertex, $y_o$ the algebraic distance to the axis of symmetry on the wing, $\tau$ the product of the time and the velocity at infinity, $W_n$ the normal displacement velocity of a point of the wing in the course of the vibration, $\zeta = y + iz$ a complex variable formed with the ordinate $y$ and the height $z$, measured normal to the wing, from a point of the flow, $f$ the complex potential of the field of the plane flow of a source of the intensity 1 at a point $y_o$ on the upper surface of a transverse section of the wing with the abscissa $x$, discharging into a sink at the same point on the lower surface. The integral is extended to the entire section of the wing which depends on the abscissa $x$.

Between the vertex of the wing and the reentrant point of the trailing edge, the calculation is identical with the one made for the delta wing, and we plan, essentially, to define the function $f(\zeta,y_o)$ in the transverse sections which intersect the trailing edge.

Since the problem is linear, we shall distinguish the symmetrical vibrations for which $W_n$ is an even function of $y_o$ from the antisymmetrical vibrations for which $W_n$ is an odd function of $y_o$. This permits limiting the integration to a half wing, replacing the function $f(\zeta,y_o)$ by $f(\zeta,y_o) + f(\zeta,-y_o)$ for the symmetrical vibrations, and by $f(\zeta,y_o) - f(\zeta,-y_o)$ for the antisymmetrical vibrations. In order not to complicate the notation, the $f$ function modified in this manner will be represented by the same letter.

Let us treat first the symmetrical problem and neglect, to begin with, the influence of the vortex sheet which is shed by the trailing edge. The field of the $f$ function is that of two sources at symmetrical points of the upper surface discharging into two sinks at the same points, but on the lower surface (fig. 55). The intensities are all equal to unity. The condition of Joukowsky imposes the position of the stagnation point at B, the image of the trailing edge.

In accordance with a general method for the problems of flow about two segments of straight lines, we shall carry out a conformal transformation defined by the field of circulation about the segments AB, A'B' (fig. 56)

$$\zeta = b\ \mathrm{sn}\left(u_1, \frac{b}{a}\right)$$

where $a$ and $b$ are functions of the abscissa $x$ which are linear in the particular case where the leading edge and the trailing edge are both straight lines.

In order to avoid using imaginary values of $u_1$ on the wing, we shall make the change

$$i\left(K_1' - u\right) = u_1 - K_1$$

The field $\zeta(u)$ is equally represented by the figure 56, but one must interchange the equipotentials and the stream lines.

$$\zeta = a\ \mathrm{dn}(u,k) \qquad k^2 = 1 - \frac{b^2}{a^2}$$

In the plane of the variable $u$, the representative field of the $f$ function can be graphically constructed (fig. 57) with sufficient precision to determine the general appearance and the singularities. The rectangle ABOI corresponds to the quarter of the right lower plane of figure 56 where the $f$ function admits as a singularity that of a sink at M where $u = u_o$.

The analytical extension by symmetries defines the complementary singularities, and the f function is determined, except for one linear function of u, corresponding to a uniform flow parallel to the real axis, by

$$f = -\frac{1}{\pi} \ln\left[\frac{\theta\left(\frac{u - u_0}{2K}\right)}{\theta\left(\frac{u + u_0}{2K}\right)}\right] + C_0 + C_1 u$$

$$\frac{df}{du} = -\frac{1}{2K\pi}\left[Z\left(\frac{u - u_0}{2K}\right) - Z\left(\frac{u + u_0}{2K}\right)\right] + C_1$$

The Joukowsky condition at the trailing edge imposes that $df/du$ be zero for $u = K$

$$0 = -\frac{1}{2K\pi}\left[Z\left(\frac{1}{2} - \frac{u_0}{2K}\right) - Z\left(\frac{1}{2} + \frac{u_0}{2K}\right)\right] + C_1 = \frac{1}{K\pi} Z_1\left(\frac{u_0}{2K}\right) + C_1$$

and consequently

$$\pi f = \ln\left[\frac{\theta\left(\frac{u + u_0}{2K}\right)}{\theta\left(\frac{u - u_0}{2K}\right)}\right] - \frac{u}{K} Z_1\left(\frac{u_0}{2K}\right) + \pi C_0$$

The constant $C_0$ may be defined by

$$C_0 = i\frac{u_0}{K} + \frac{iK'}{\pi K} Z_1\left(\frac{u_0}{2K}\right)$$

so that f will be zero when $\zeta$ is infinite and $u = iK'$.

However, this purely imaginary value may be neglected.

The problem is therefore solved by the parametric definition of the function $f(\zeta)$ with the aid of $f(u)$ and of $\zeta(u)$.

Let us now study the influence of the vortex sheet originating at the trailing edge.

Because of the linear character of the problem, the field induced by the vortices may be superimposed on the field calculated above under the condition that this induced field is not altered by the velocity which is normal to the wing. This reservation will be respected if the field induced by each pair of symmetrical vortices of the sheet encloses the mean position of the wing (fig. 58) and satisfies the Joukowsky condition at the trailing edge.

The field of these symmetrical vortices may be transposed in the plane of the variable $u$ (fig. 59) where it appears like the field of vortices with a uniform flow, parallel to the real axis, placing the stagnation point at the image B of the trailing edge. The corresponding function $f_1(u)$ is determined by its singularities and their images of the analytic continuation by symmetry. For a vortex with a circulation equal to one

$$f_1 = \frac{i}{2\pi} \ln\left[\frac{\theta_2\left(\frac{u + i\alpha}{2K}\right)}{\theta_2\left(\frac{u - i\alpha}{2K}\right)}\right] + D_o + D_1 u$$

$$\frac{df_1}{du} = \frac{i}{4\pi K}\left[Z_2\left(\frac{u + i\alpha}{2K}\right) - Z_2\left(\frac{u - i\alpha}{2K}\right)\right] + D_1$$

The stagnation point is at B if $df_1/du$ is zero for $u = K$.

$$0 = \frac{i}{2\pi K} Z_3\left(\frac{i\alpha}{2K}\right) + D_1 = \frac{1}{2\pi K'} Z_1\left(\frac{\alpha}{2K'}, q'\right) + \frac{\alpha}{2KK'} + D_1$$

where the logarithmic derivatives of the relations established at the end of 3.1 have been utilized for the calculation of the $Z_3$ function for a purely imaginary argument.

To facilitate the notation, the $Z_1$ function will be represented by

$$Z_1\left(\frac{\alpha}{2K'}, q'\right) = Z_1'\left(\frac{\alpha}{2K'}\right)$$

the $f_1$ function is then defined by

$$2\pi f_1 = i \ln\left[\frac{\theta_2\left(\frac{u + i\alpha}{2K}\right)}{\theta_2\left(\frac{u - i\alpha}{2K}\right)}\right] - \frac{u}{K'}\left[Z_1'\left(\frac{\alpha}{2K'}\right) + \frac{\pi\alpha}{K}\right] + 2\pi D_o$$

The constant $D_o$ can be determined by the condition that $f_1$ should be zero for infinite $\zeta$, that is, for $u = iK'$.

Nevertheless, it appears as purely imaginary and may be neglected.

The ordinate $y_o$ of the vortex center is

$$y_o = a\ \mathrm{dn}(K + iK' - i\alpha, k)$$

$$= b\ \mathrm{sn}(\alpha, k') = b\ \mathrm{sn}'\alpha$$

The equilibrium of pressures on both sides of the vortex sheet requires that the potential difference be invariable along a vortex at a point which is displaced at the speed of the flow.

This variation of potential, equal to the circulation on a contour starting from a point of the sheet and coming back to it, avoiding the wing and the sheet, will be the weighted sum of the variations of the $f$ and $f_1$ functions.

When such a contour does not enclose a vortex, the theta functions are uniform and the variation of $u$ is $2K$. Consequently, the variations of the functions are, respectively

$$\Delta f = -\frac{2}{\pi} Z_1\left(\frac{u_o}{2K}\right)$$

$$\Delta f_1 = -\frac{1}{\pi}\left[Z_1'\left(\frac{\alpha}{2K'}\right) + \frac{\pi\alpha}{K}\right]$$

If the image of the contour in the plane of the variable $u$ encloses the vortex $\frac{i}{\pi} \ln\left[\theta_2\left(\frac{u-\alpha}{2K}\right)\right]$, it is convenient to add 1 to $\Delta f_1$.

The potential is the real part of

$$\Phi = \int_o^K W_n(u_o,x,\tau)f(u,u_o,k)ak^2 \text{sn } u_o \text{ cn } u_o \, du_o +$$

$$\int_o^K \gamma f_1(u,\alpha,k)b \text{ cn } \alpha \text{ dn } \alpha \, d\alpha$$

if $\gamma \, dy_o$ is the intensity of the vortex of the magnitude $dy_o = b \text{ cn } \alpha \text{ dn } \alpha \, d\alpha$.

The circulation or variation of the potential on the above contour is

$$\Gamma = -\frac{2}{\pi}\int_o^K W_n(u_o,x,\tau)Z_1\left(\frac{u_o}{2K}\right)ak^2 \text{sn } u_o \text{ cn } u_o \, du_o -$$

$$\frac{1}{\pi}\int_o^K \gamma\left[Z_1'\left(\frac{\alpha}{2K'}\right) + \frac{\pi\alpha}{K}\right]b \text{ cn } \alpha \text{ dn } \alpha \, d\alpha + \int_{\alpha_o}^{iK} \gamma b \text{ cn } \alpha \text{ dn } \alpha \, d\alpha$$

where $\alpha_o$ is the image of the point – with the ordinate $y_o$ – where the vortex sheet intersects the contour.

The condition of constancy of the circulation for a point entrained by the flow is written

$$0 = \frac{d\Gamma}{d\tau} \sim \frac{\partial\Gamma}{\partial\tau} + \frac{\partial\Gamma}{\partial x}$$

and, consequently, $\Gamma$ has the form $\Gamma(x - \tau, y_o)$.

The two first integrals do not depend on $y_o$ and the latter may be written

$$\int_{y_o}^{b} \gamma \cdot dy_o$$

Consequently

$$\gamma = -\frac{d}{dy_o}\left[\Gamma(x - \tau, y_o)\right]$$

This relation utilizes completely the law of the variation of $\Gamma$ with $y_o$ and it suffices to retain the expression of $\Gamma$ for $y_o = b$ in order to eliminate the last integral

$$\Gamma(x - \tau, b) = -\frac{2}{\pi}\int_{o}^{K} W_n(u_o, x, \tau) Z_1\left(\frac{u_o}{2K}\right) ak^2 \operatorname{sn} u_o \operatorname{cn} u_o \, du_o +$$

$$\frac{1}{\pi}\int_{o}^{b} \frac{d}{dy}\left[\Gamma(x - \tau, y)\right]\left[Z_1\left(\frac{\alpha}{2K}\right) + \frac{\pi\alpha}{K}\right] dy$$

This integro-differential equation in $\Gamma$ which can, besides, be transformed into an integral equation by an integration by parts of the last integral, is not very manageable in the general case, but for the applications usually undertaken, the vibrations are harmonic, and the solution $\Gamma$ is likewise harmonic if the period of establishment of the motion is neglected. One agrees to retain only the real parts

$$W_n = W_o(y, x)e^{i\omega\tau} = W_o\left[a \operatorname{dn} u_o, x\right]e^{i\omega\tau} \qquad \Gamma = \Gamma_o(y)e^{i\omega(\tau - x)}$$

and the integral equation assumes the form

$$\Gamma_o(b) = -2\frac{e^{i\omega x}}{\pi}\int_0^K W_o(u_o,x)Z_1\left(\frac{u_o}{2K}\right)ak^2 \text{sn } u_o \text{ cn } u_o \, du_o +$$

$$\frac{1}{\pi}\int_0^b \frac{d}{dy}\left[\Gamma_o(y)\right]\left[Z_1'\left(\frac{\alpha}{2K'}\right) + \frac{\pi\alpha}{K}\right]dy$$

where

$$y = b \text{ sn}' \alpha$$

and $a$, $b$, $k^2$, $K$, $K'$ are known functions of $x$, the two first ones linear for rectilinear leading and trailing edges.

If the last integral is integrated by parts, it takes the form

$$\frac{1}{\pi}\left[\left[\Gamma_o(b \text{ sn}' \alpha) - \Gamma_o(b)\right]\left[Z_1'\left(\frac{\alpha}{2K'}\right) + \frac{\pi\alpha}{K}\right]\right]_0^K -$$

$$\frac{1}{\pi}\int_0^K \left[\Gamma_o(b \text{ sn}' \alpha) - \Gamma_o(b)\right]\frac{d}{d\alpha}\left[Z_1'\left(\frac{\alpha}{2K'}\right) + \frac{\pi\alpha}{K}\right]d\alpha$$

The integrated part is zero because $Z_1'$ is zero for $\alpha = 0$ and behaves like $\frac{2K'}{\alpha - K'}$ in the neighborhood of $\alpha - K'$ whereas, if there exists no singularity of the wing contour rendering the derivative of $\Gamma$ infinite, the first term behaves like

$$-\frac{d\Gamma_o}{db}\frac{d(b \text{ sn}' \alpha)}{d\alpha}(\alpha - K') = \frac{d\Gamma_o}{db} b \text{ cn}' \alpha \text{ dn}' \alpha(K' - \alpha)$$

$\text{cn}' \alpha$ tends toward zero when $\alpha$ tends toward $K'$ and the product of the two terms tends toward zero. The argument shows, moreover, that the integral which remains is finite.

If one notes, on the other hand, that

$$\frac{d}{d\alpha}\left[Z_1'\left(\frac{\alpha}{2K'}\right)\right] = \frac{d}{d\alpha}\left[Z_3'\left(\frac{\alpha - K' - iK}{2K'}\right)\right]$$

$$= 2K' \text{ dn}'^2(\alpha - K' - iK) - 2E'$$

$$= 2k^2K' \frac{\text{sn}'^2\alpha}{\text{cn}'^2\alpha} - 2E'$$

the integral equation takes the form

$$\Gamma_o(b) = -\frac{2e^{i\omega x}}{\pi}\int_o^K W_o(u_o,x)Z_1\left(\frac{u_o}{2K}\right)ak^2 \operatorname{sn} u_o \operatorname{cn} u_o \, du_o -$$

$$\frac{1}{\pi}\int_o^{K'} \left[\Gamma_o(b \operatorname{sn} \alpha) - \Gamma_o(b)\right]\left[-2k^2K' \frac{\operatorname{sn'}^2\alpha}{\operatorname{cn'}^2\alpha} - 2E' + \frac{\pi}{K}\right]d\alpha$$

This equation may be differentiated in order to furnish $\frac{d\Gamma_o}{db}(b)$ as a function of the values of $\Gamma_o$ from 0 to b. It permits therefore the construction of $\Gamma_o(b)$ by graphic integration for a form of harmonic vibration given by $W_n(u_o,x)$. For practical purposes, it is more convenient to retain the equation in its first form.

For the transverse cross section of the abscissa $x_o$ passing through the point upstream from the trailing edge and constituting the limit of the delta upstream giving rise to an easy calculation, the formula furnishes $\Gamma(0)$ for decomposed elliptic functions and zeta functions corresponding to $k = 1$. Besides, the quantity $\Gamma(0)$ is in that case the circulation for a contour passing through the axis of the delta wing. Starting from this initial value, the value $\Gamma(b)$ may be calculated step by step in successive cross sections the abscissa $x$ of which varies from $x_o$ up to the abscissa of the wing tips. If an effect $\Gamma(b)$ is known from 0 to $b(x)$ in a cross section of the abscissa $x$, the curve may be extrapolated by a segment of a straight line of undetermined slope $d\Gamma/db$. The integral equation, written in the cross section $x + \delta x$ where $\delta x$ is a finite variation is then an equation of the first degree in $d\Gamma/db$. The procedure, equivalent to the differentiation of the integral equation, has the advantage of avoiding the calculation of the derivatives of the elliptic functions and of the zeta functions which depend on the abscissa $x$ by the intermediary of their modulus $k^2$.

When $\Gamma(b)$ and its derivative $d\Gamma/db$ are calculated, the potential is completely determined by the distribution of intensity of the circulation $\gamma$ linked to $d\Gamma/db$. The calculation of the aerodynamic forces may then be carried out as for the delta wing.

If one has to deal with antisymmetrical vibrations, the calculation is parallel and it will be sufficient in this case to perform again the determination of the functions $f$ and $f_1$.

Figure 60 represents the field of the function $f(\zeta)$, and figure 61 its image in the plane of the variable $u$ corresponding to the function $f(u)$ which is defined by its singularities, that is, that of a sink at M and that of a vortex with the image I of infinity.

$$f = -\frac{1}{\pi} \ln\left[\frac{\theta\left(\frac{u - u_0}{4K}\right)\theta_3\left(\frac{u + u_0}{4K}\right)}{\theta_3\left(\frac{u - u_0}{4K}\right)\theta\left(\frac{u + u_0}{4K}\right)}\right] - i\,A \ln\left[\frac{\theta\left(\frac{u - iK'}{4K}\right)\theta_1\left(\frac{u - iK'}{4K}\right)}{\theta\left(\frac{u + iK'}{4K}\right)\theta_1\left(\frac{u + iK'}{4K}\right)}\right] + C_0 + C_1 u$$

$$= -\frac{1}{2\pi} \ln\left[\frac{\operatorname{sn}\left(\frac{u - u_0}{2}\right)}{\operatorname{sn}\left(\frac{u + u_0}{2}\right)}\right] - A\varphi(u) + C_0 + C_1 u$$

$$\frac{df}{du} = -\frac{1}{2\pi}\,\frac{\operatorname{cn}\left(\frac{u - u_0}{2}\right)\operatorname{dn}\left(\frac{u - u_0}{2}\right)}{\operatorname{sn}\left(\frac{u - u_0}{2}\right)} + \frac{1}{2\pi}\,\frac{\operatorname{cn}\frac{u + u_0}{2}\operatorname{dn}\frac{u + u_0}{2}}{\operatorname{sn}\frac{u + u_0}{2}} - A \operatorname{dn} u + C_1$$

The constants $A$ and $C$ are to be determined in such a manner that stagnation points exist at $B$ and $O$ in the plane of the variable $u$.

Let us study likewise the function $f_1(\zeta)$ characterizing the vortex sheet (fig. 62), constructing its image in the plane of the variable $u$ (fig. 63).

$$f_1 = \frac{i}{2\pi} \ln\left[\frac{\theta\left(\frac{u - iK' + i\alpha}{4K}\right)\theta_1\left(\frac{u - iK' + i\alpha}{4K}\right)}{\theta\left(\frac{u + iK' - i\alpha}{4K}\right)\theta_1\left(\frac{u + iK' - i\alpha}{4K}\right)}\right] + iB \ln\left[\frac{\theta\left(\frac{u - iK'}{4K}\right)\theta_1\left(\frac{u - iK'}{4K}\right)}{\theta\left(\frac{u + iK'}{4K}\right)\theta_1\left(\frac{u + iK'}{4K}\right)}\right] + D_0 + D_1 u$$

$$= \frac{i}{2\pi} \ln\left[\frac{\operatorname{sn}'\frac{u - K - iK' + i\alpha}{2}}{\operatorname{sn}'\frac{u - K - iK' - i\alpha}{2}}\right] + B\varphi(u) + D_0 + D_1 u$$

$$\frac{df_1}{du} = \frac{i}{4\pi}\,\frac{\operatorname{cn}'\left(\frac{u - K - iK' + i\alpha}{2}\right)\operatorname{dn}'\left(\frac{u - K - iK' + i\alpha}{2}\right)}{\operatorname{sn}'\left(\frac{u - K - iK' + i\alpha}{2}\right)} -$$

$$\frac{i}{2\pi}\,\frac{\operatorname{cn}'\left(\frac{u - K - iK' - i\alpha}{2}\right)\operatorname{dn}'\left(\frac{u - K - iK' - i\alpha}{2}\right)}{\operatorname{sn}'\left(\frac{u - K - iK' - i\alpha}{2}\right)} + B \operatorname{dn} u + D_1$$

The constants $B$ and $D_1$ are to be determined in such a manner that $df_1/du$ is zero at $B$ and $O$.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

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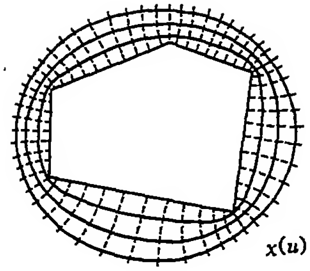


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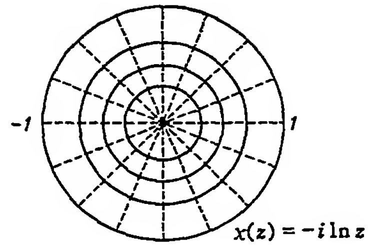


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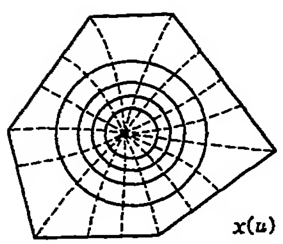


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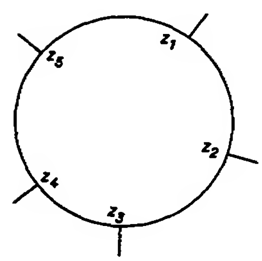


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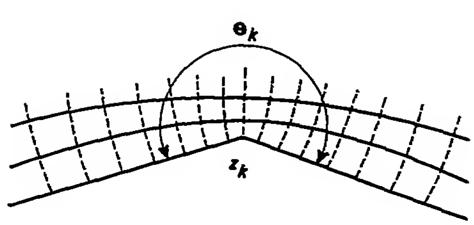


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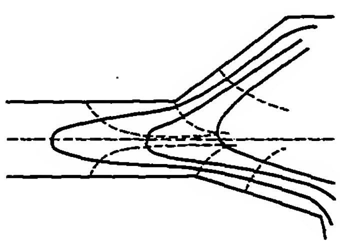

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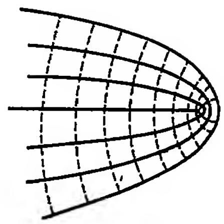

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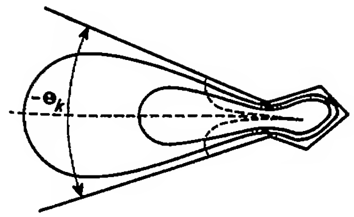


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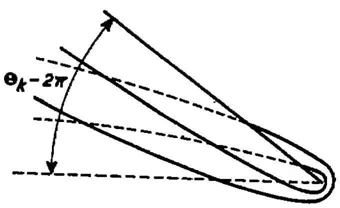


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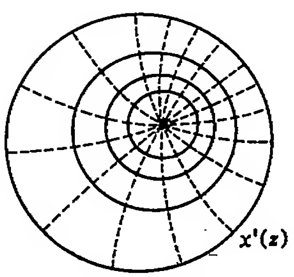


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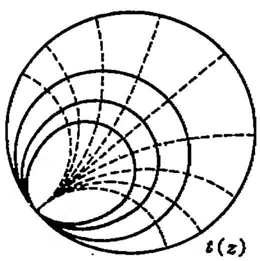


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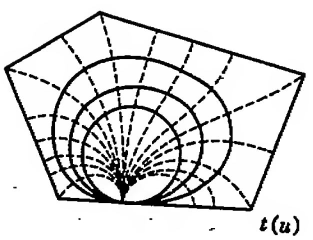


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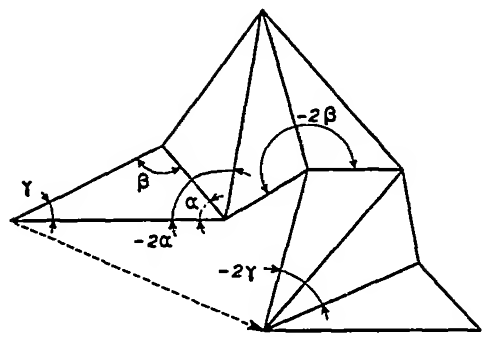


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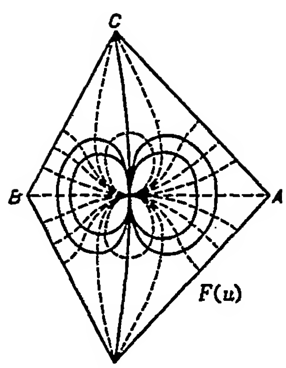


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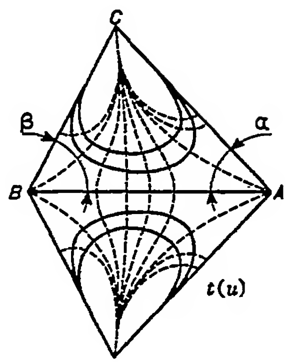


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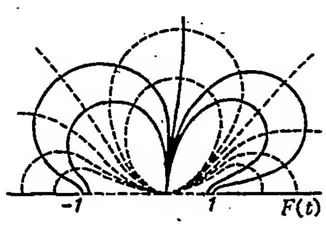


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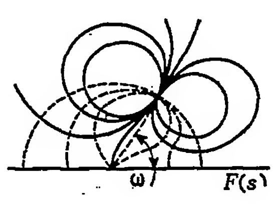


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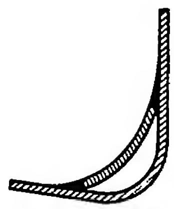

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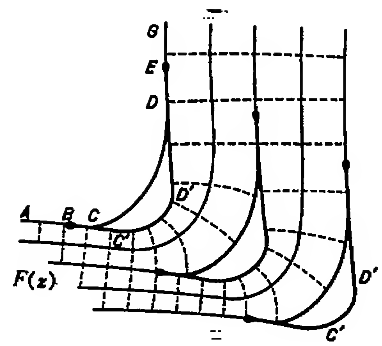


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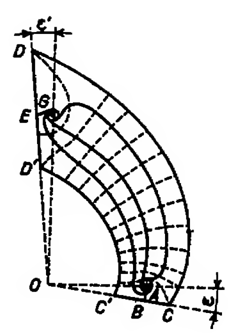


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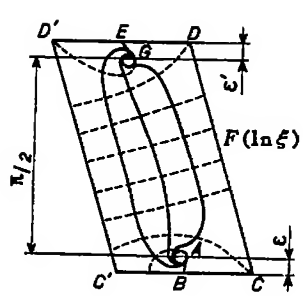


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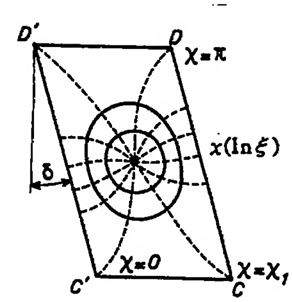


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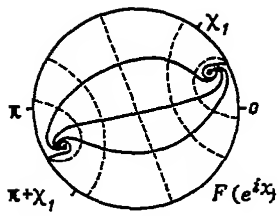


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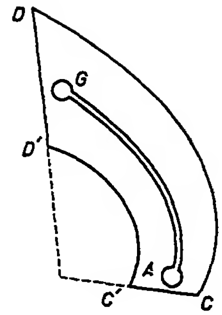


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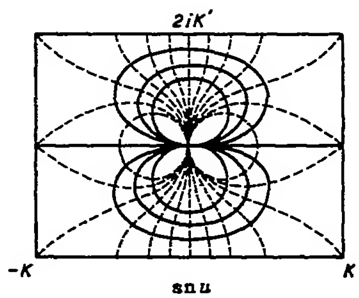


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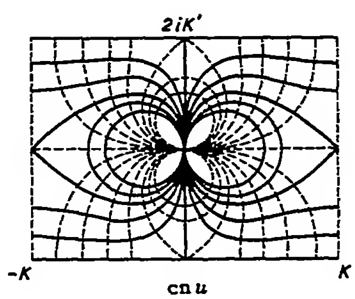


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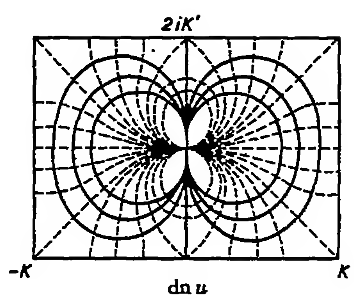


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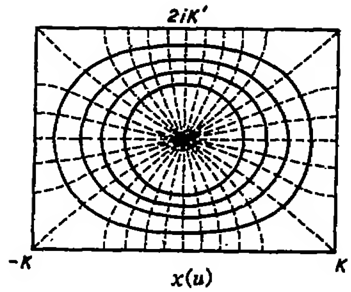


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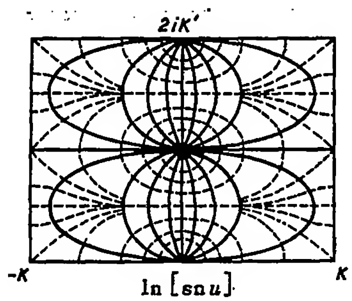


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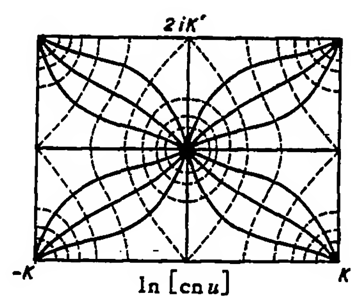


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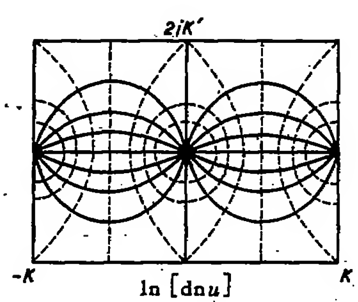


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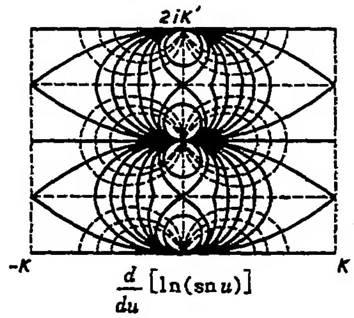


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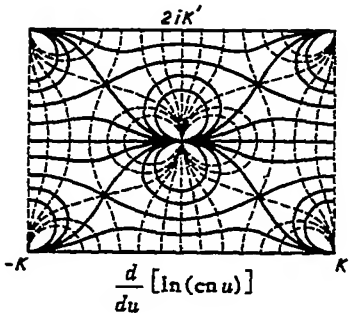


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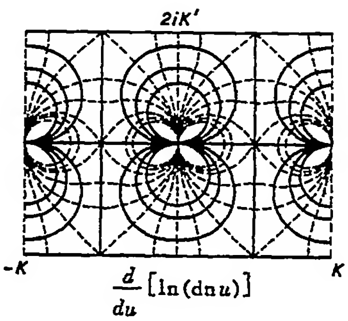


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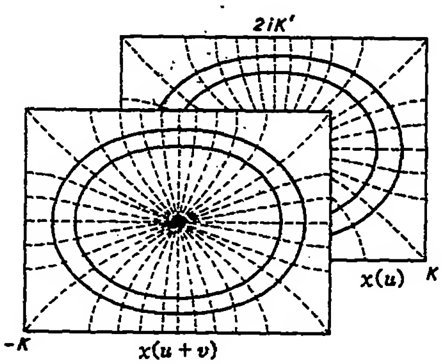


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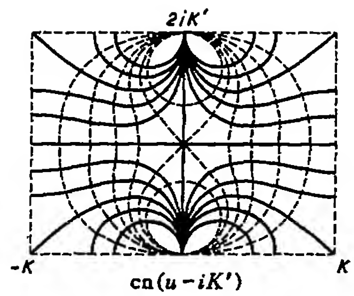


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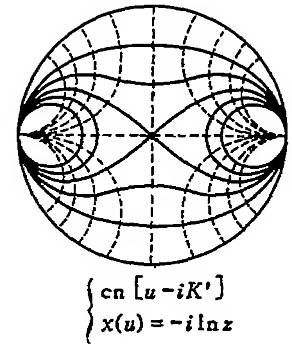


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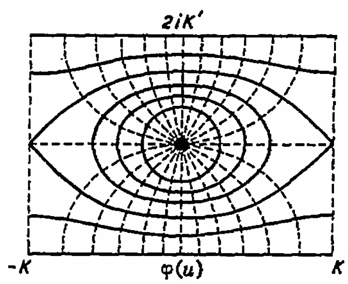


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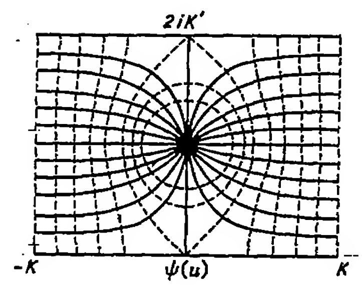


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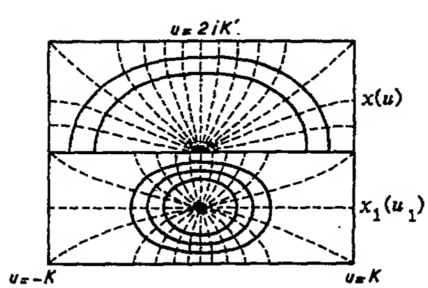


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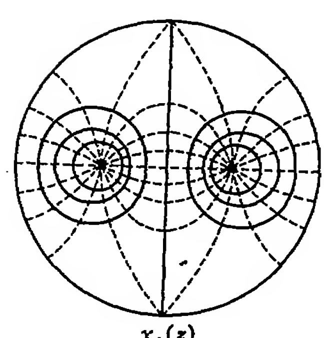


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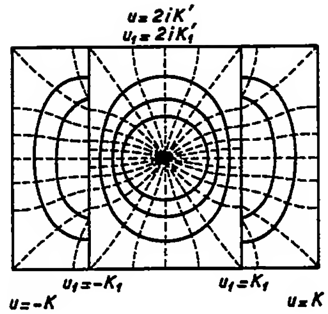


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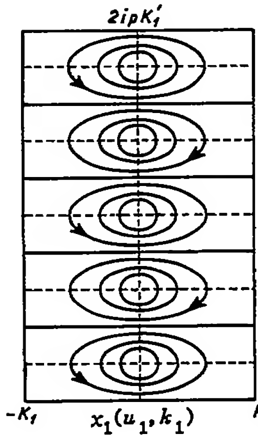


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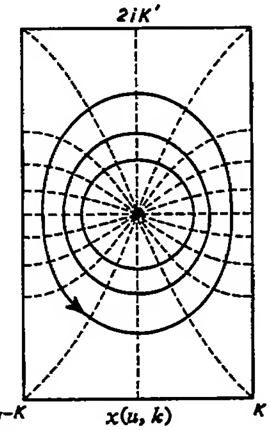


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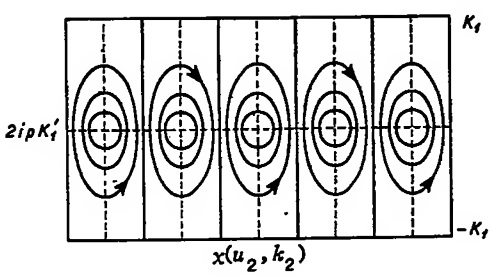


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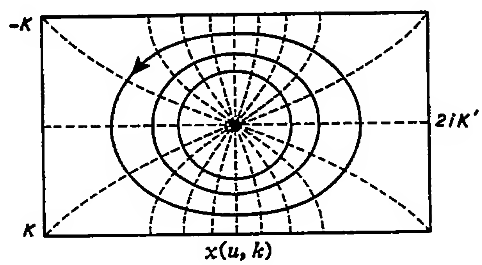


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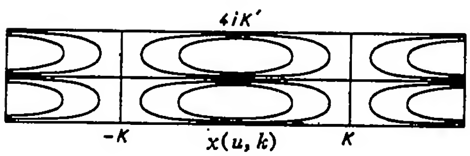


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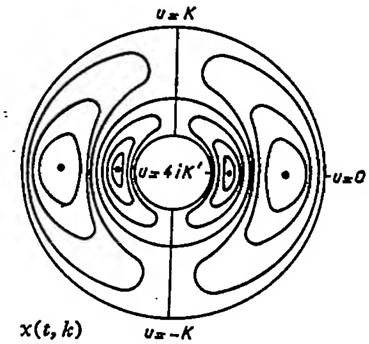


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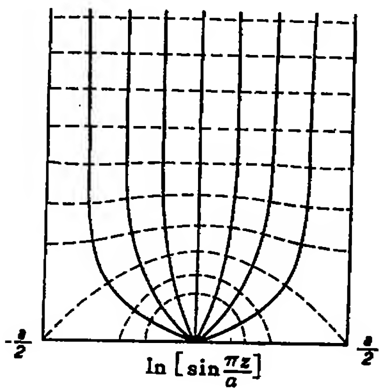


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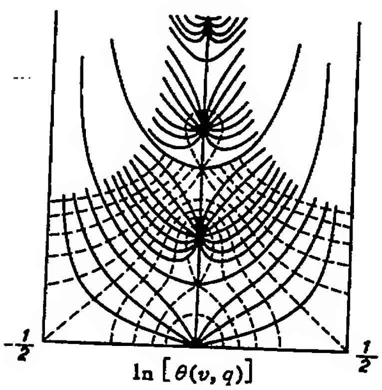


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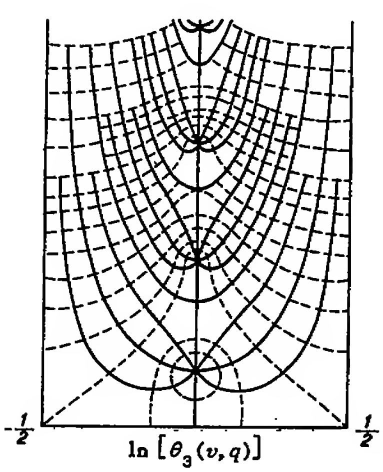


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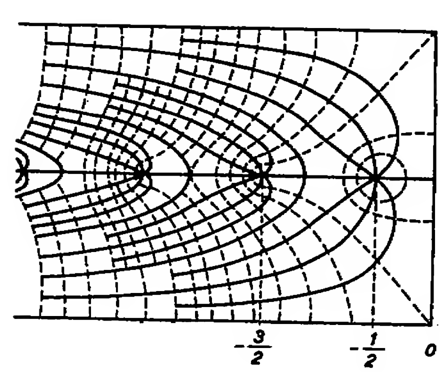


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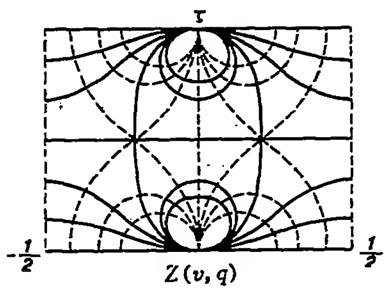


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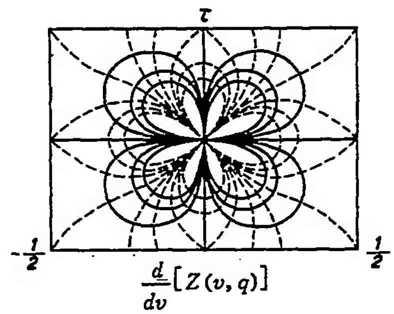


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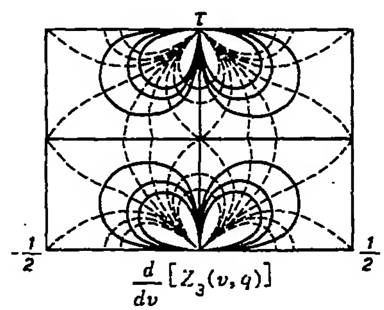


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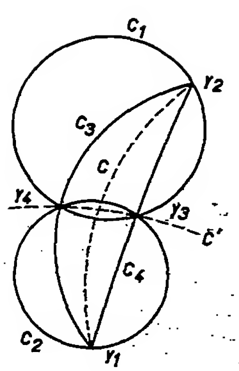
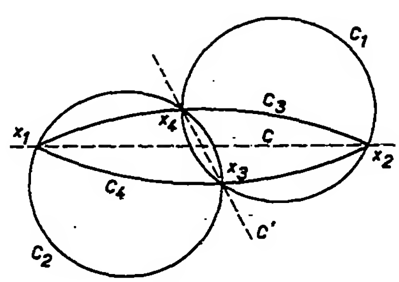



Figure 38(a).

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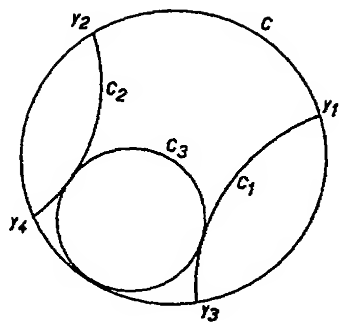


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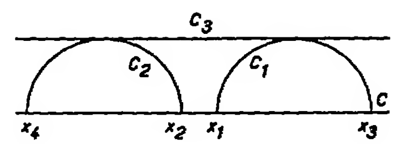


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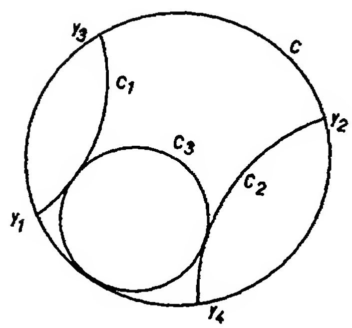


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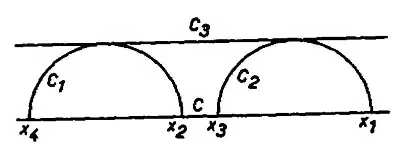


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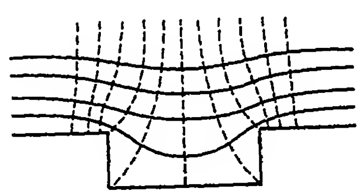

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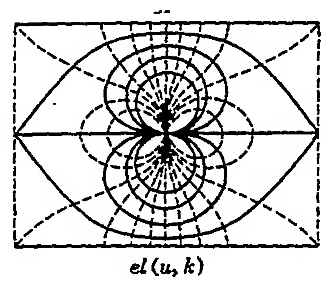



Figure 42.

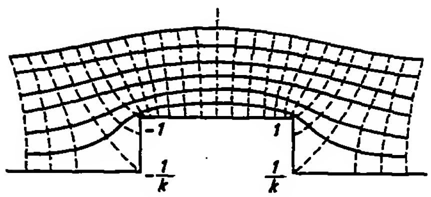
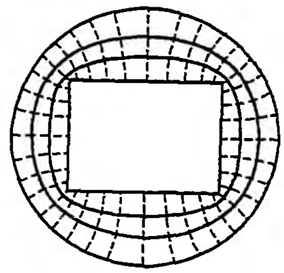



Figure 43.



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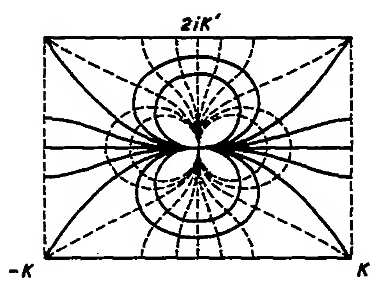


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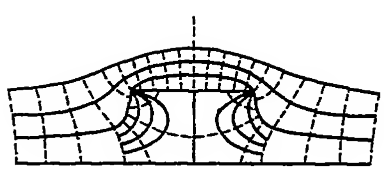


Figure 46.

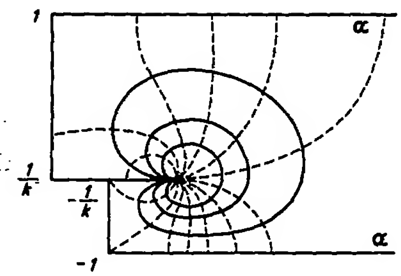



Figure 47(a).

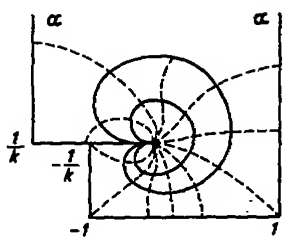



Figure 47(b).

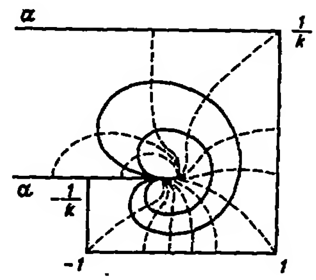



Figure 47(c).

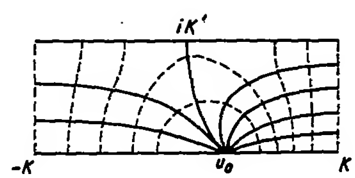



Figure 48(a).

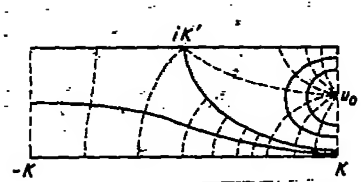



Figure 48(b).

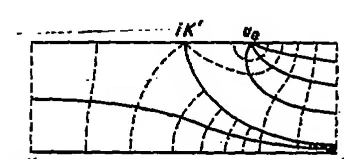



Figure 48(c).

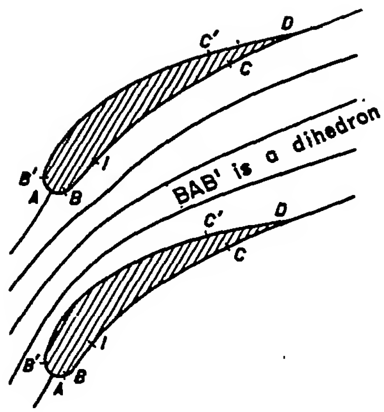



Figure 49.

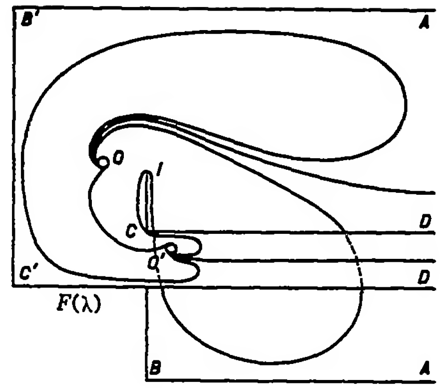



Figure 50.

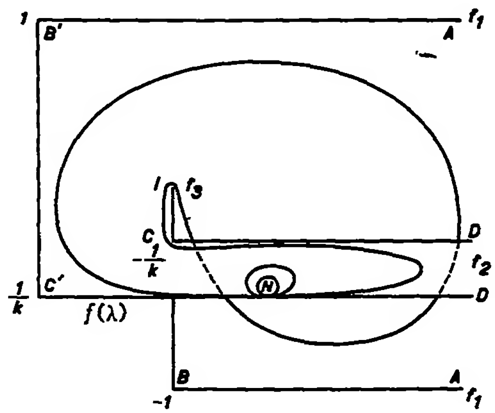



Figure 51.

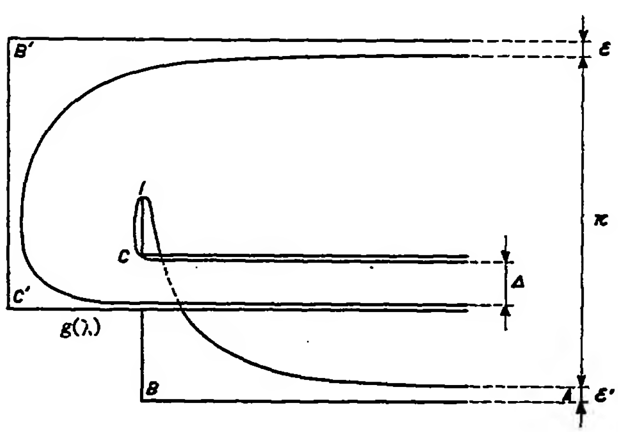



Figure 52.

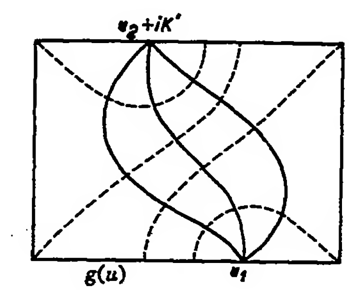
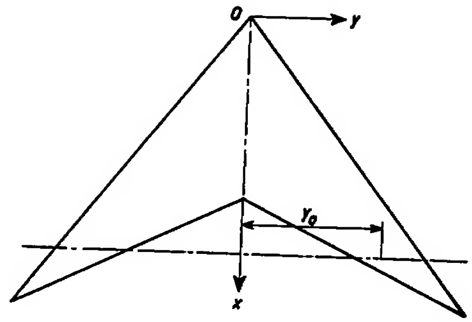



Figure 53.

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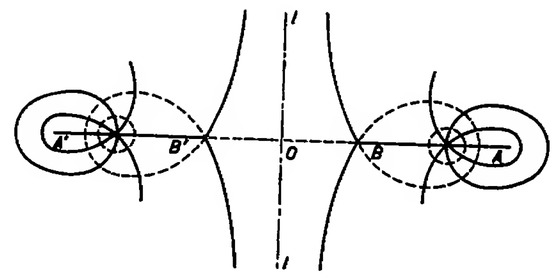


Figure 55.

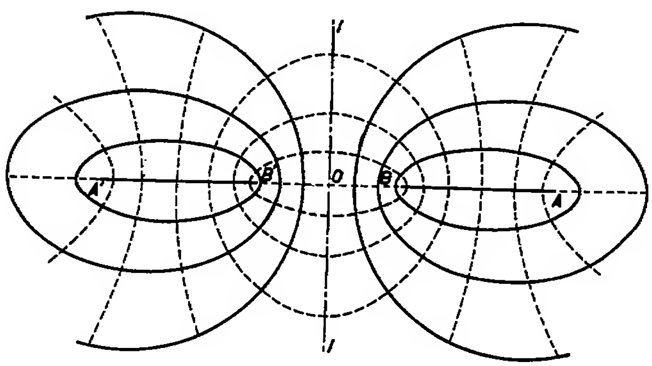


Figure 56.

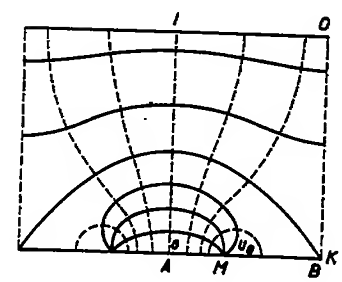


Figure 57.

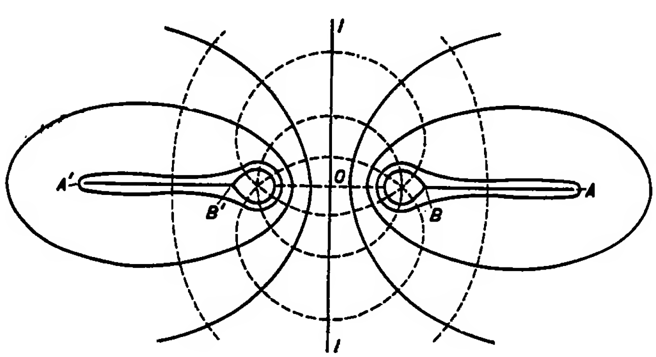


Figure 58.

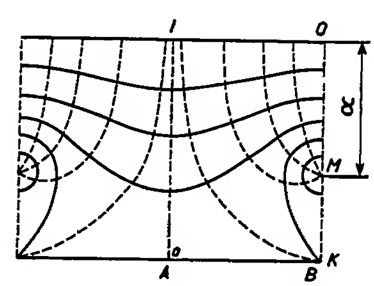


Figure 59.

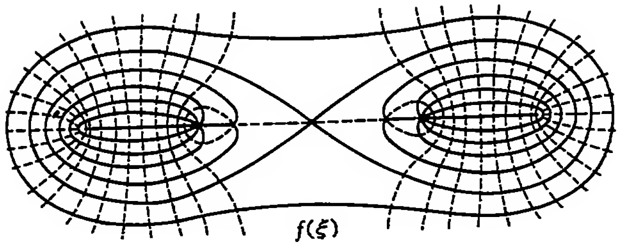


Figure 60.

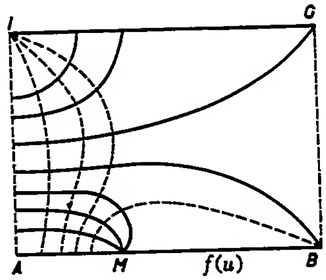


Figure 61.

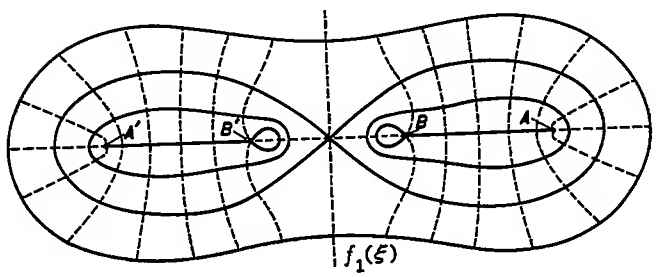


Figure 62.

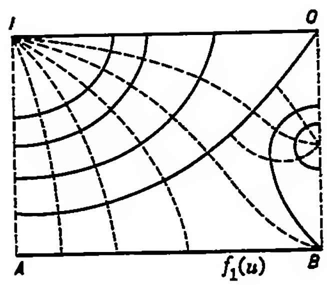


Figure 63.